Incompressibility Pressure Computation

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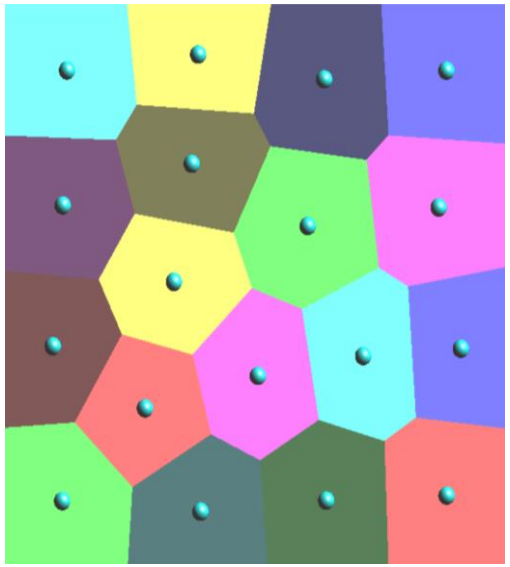
Incompressibility

- Is essential for a realistic fluid behavior
  - Less than 0.1% density deviation in typical scenarios
- Inappropriate compression leads to volume oscillations or volume loss
- Significant influence on the performance
  - Local approaches require small time steps
  - Global approaches work with large time steps

Fluid Simulation Setting

Fluid



Set of parcels

$\boldsymbol{x}$

$\boldsymbol{v}$

Positions and velocities of parcels over time

Governing Equations

- Position change: $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$
- Velocity change: $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}$
- Incompressibility constraint: $\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \cdot \nabla \cdot \mathbf{v} = 0$

Accelerations

- Realize fluid properties
- Navier-Stokes equation

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \underbrace{-\frac{1}{\rho}\nabla p}_{\mathbf{a}^{\mathrm{p}}} \underbrace{+\nu\nabla^2\mathbf{v} + \mathbf{g}}_{\mathbf{a}^{\mathrm{nonp}}}$$

Incompressibility Viscosity Gravity

Pressure Acceleration

- Realizes the incompressibility constraint
- Typical implementation
  - Predict velocity from non-pressure accelerations

    $$\mathbf{v}^* = \mathbf{v} + \Delta t \cdot \mathbf{a}^{\mathrm{nonp}}$$

  - Compute pressure p such that the velocity change $\Delta t \cdot \mathbf{a}^{\mathrm{p}}$ results in a divergence-free velocity field, i.e.

    $$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \cdot \nabla \cdot \mathbf{v} = 0$$

Pressure

- Quantifies compression
- Grows with growing density error / deviation
- Pressure acceleration $-\frac{1}{\rho}\nabla p$
  - Accelerates particles from positions with larger pressure / density error towards positions with smaller pressure

Compression

- Current density deviation: $\rho - \rho^0$
- Predicted density deviation:

$$\mathbf{v}^* = \mathbf{v} + \Delta t \cdot \mathbf{a}^{\text{nonp}} \Rightarrow \mathbf{x}^* \Rightarrow \rho^* \Rightarrow \rho^* - \rho^0$$

$$\Rightarrow \rho^* = \rho - \Delta t \cdot \rho^0 \cdot \nabla \cdot \mathbf{v}^* \Rightarrow \rho^* - \rho^0$$

- Density change within a time step
  - Velocity divergence: $-\Delta t \cdot \rho^0 \cdot \nabla \cdot \mathbf{v}$
  - Divergence of the predicted velocity: $-\Delta t \cdot \rho^0 \cdot \nabla \cdot \mathbf{v}^*$

Local Pressure

– Per particle *i* with a state equation

$$p_i = k(\rho_i^* - \rho^0) \qquad p_i = k_1(\rho_i - \rho^0)^{k_2}$$

$$p_i = k(\rho_i - \rho^0) \qquad p_i = -k(\Delta t \cdot \rho^0 \cdot \nabla \cdot \mathbf{v}_i^*) \qquad \dots$$

$$p_i = -k(\Delta t \cdot \rho^0 \cdot \nabla \cdot \mathbf{v}_i) \qquad p_i = k_i(\frac{\rho_i}{\rho^0} - 1)$$

– Typically clamped to non-negative values

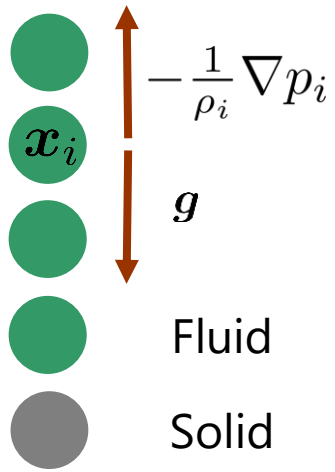
– Very efficient per-particle computations $p_i \Rightarrow -\frac{1}{\rho_i}\nabla p_i$

State Equation Governs Density Error

- A fluid under gravity at rest
  - Gravity cancels pressure acceleration

$$\boldsymbol{g} = -\boldsymbol{a}_i^{\mathrm{p}} = \frac{1}{\rho_i}\nabla p_i = \sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla W_{ij}$$

$$= \sum_j m_j \left(\frac{k(\rho_i - \rho_0)}{\rho_i^2} + \frac{k(\rho_j - \rho_0)}{\rho_j^2}\right)\nabla W_{ij}$$



  - Differences between p_i and p_j are independent from k
  - Smaller k results in larger density error $\rho_i - \rho_0$ to get the desired pressure

Local vs. Global Pressure

- Local
  - Pressure per particle
  - Weakly compressible
- Global
  - Pressure per particle
  - Conceptually considers the pressure gradient
  - Formulations involve pressure at adjacent particles
  - Linear system
  - Incompressible

Global Pressure – An Intuition

- Pressure field corresponds to

  an acceleration $$\mathbf{a}_i^{\mathrm{p}} = -\frac{1}{\rho_i}\nabla p_i$$

  a velocity change $$\Delta\mathbf{v}_i^{\mathrm{p}} = -\Delta t\frac{1}{\rho_i}\nabla p_i$$

  with a divergence $$\nabla\cdot\Delta\mathbf{v}_i^{\mathrm{p}} = -\nabla\cdot(\Delta t\frac{1}{\rho_i}\nabla p_i)$$

  and a density change $$\Delta\rho_i^{\mathrm{p}} = \Delta t\cdot\rho_i\cdot\nabla\cdot(\Delta t\frac{1}{\rho_i}\nabla p_i)$$

- Density change due to pressure gradient cancels compression $$\underbrace{\Delta t\cdot\rho_i\cdot\nabla\cdot(\Delta t\frac{1}{\rho_i}\nabla p_i)}_{\text{Density change due to pressure gradient}} + \underbrace{(\rho_i^* - \rho^0)}_{\text{Compression}} = 0$$

Pressure Poisson Equation PPE

- Per-particle equation

$$\Delta t \cdot \rho_i \cdot \nabla \cdot (\Delta t \frac{1}{\rho_i} \nabla p_i) + (\rho_i^* - \rho^0) = 0$$

- ∇p_i involves pressure at adjacent particles
- Linear system / global formulation
- Various forms with different compression computations, i.e. source terms, e.g.

$$\Delta t \cdot \nabla^2 p_i = \frac{\rho^0 - \rho_i^*}{\Delta t} \qquad \Delta t^2 \cdot \nabla^2 p_i = \rho^0 - (\rho_i - \Delta t \cdot \rho_i \cdot \nabla \cdot \mathbf{v}_i^*)$$

$$\Delta t \cdot \nabla^2 p_i = \rho_i \cdot \nabla \cdot \mathbf{v}_i^*$$

SPH Discretization

- System $\mathbf{Ap} = \mathbf{s}$, particle equations $(\mathbf{Ap})_i = s_i$
- Source term s_i
  - SPH density and SPH divergence formulations
- Laplace term $(\mathbf{Ap})_i$
  - Various options, e.g. IISPH

$$\mathbf{a}_i^{\mathrm{p}} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij}$$

$$(\mathbf{Ap})_i \approx \Delta t^2 \sum_j m_j \left(\mathbf{a}_i^{\mathrm{p}} + \mathbf{a}_j^{\mathrm{p}} \right) \nabla W_{ij}$$

PPE Solver (IISPH)

- Jacobi solver
- Iterative pressure update
- Iteration l

for all *particle* i **do**

$$(\mathbf{a}_i^{\mathrm{p}})^l = -\sum_j m_j \left(\frac{p_i^l}{\rho_i^2} + \frac{p_j^l}{\rho_j^2} \right) \nabla W_{ij}$$

for all *particle* i **do**

$$(\mathbf{Ap})_i^l = \Delta t^2 \sum_j m_j \left((\mathbf{a}_i^{\mathrm{p}})^l + (\mathbf{a}_j^{\mathrm{p}})^l \right) \nabla W_{ij}$$

Solver Iteration

- Can be implemented in various forms
  - IISPH [Ihmsen et al.]
  - PCISPH [Solenthaler, Pajarola]
  - PBF [Macklin, Müller]
  - DFSPH [Bender, Koschier]
- Derivations use different intuitions
- Solver iterations are all equivalent

  [Koschier et al. “A Survey on SPH Methods in Computer Graphics”, STAR, EG 2022]

Performance Differences

- IISPH, PCISPH, PBF, DFSPH differ in terms of performance
  - Different boundary handling
  - Different implementation details, e.g. $\sum_j (\nabla W_{ij}^{\mathrm{T}}) \nabla W_{ij} + \sum_j (\sum_j \nabla W_{ij}^{\mathrm{T}}) \nabla W_{ij}$
    - Occurs in all implementations
    - Approximated for a template particle
    - Computed per particle
    - Computed per particle per iteration

Global Pressure - Summary

- IISPH, PCISPH, PBF, DFSPH
  - Jacobi PPE solver
  - Different intuitions in the derivations
  - Original formulations with different boundary handlings
  - Implementation details / SPH discretizations differ
- Typically preferred over local pressure solvers
  - Efficient due to large time steps
  - Robust / stable
  - Easy and intuitive to parameterize

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