A Survey on SPH Methods in Computer Graphics

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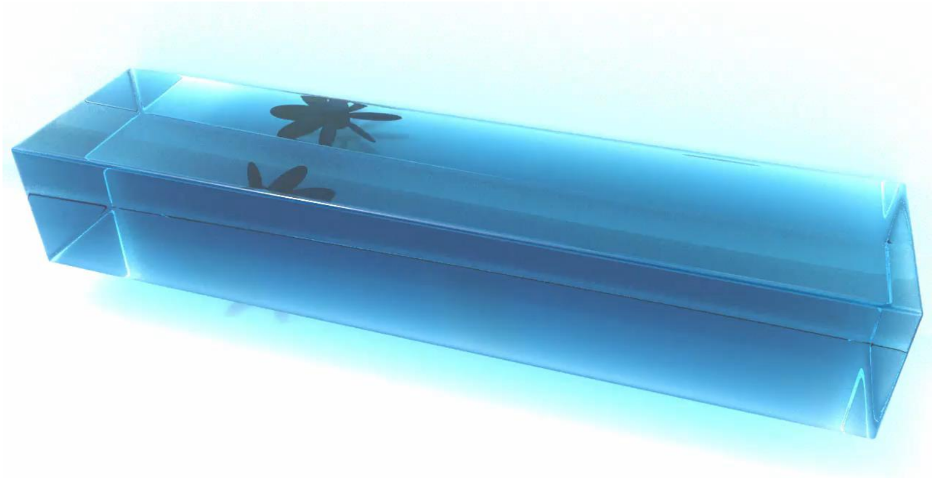
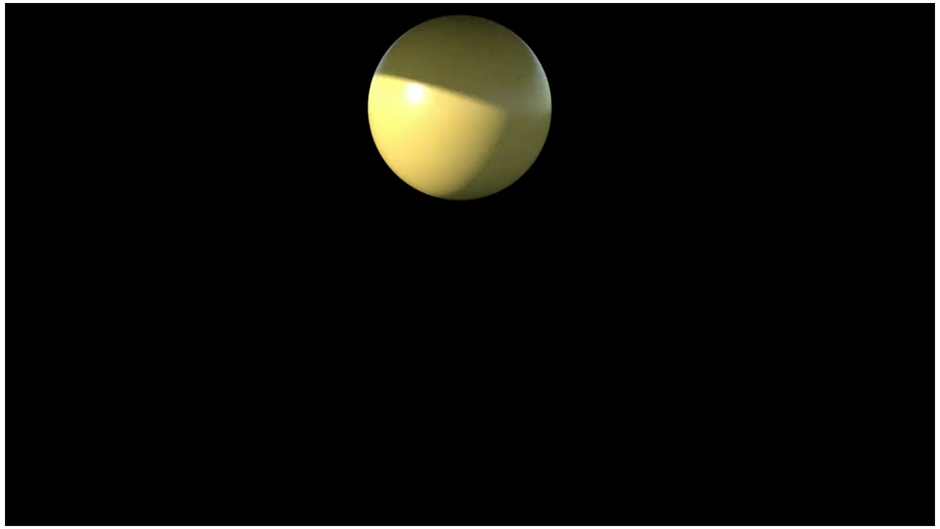
State-of-the-Art

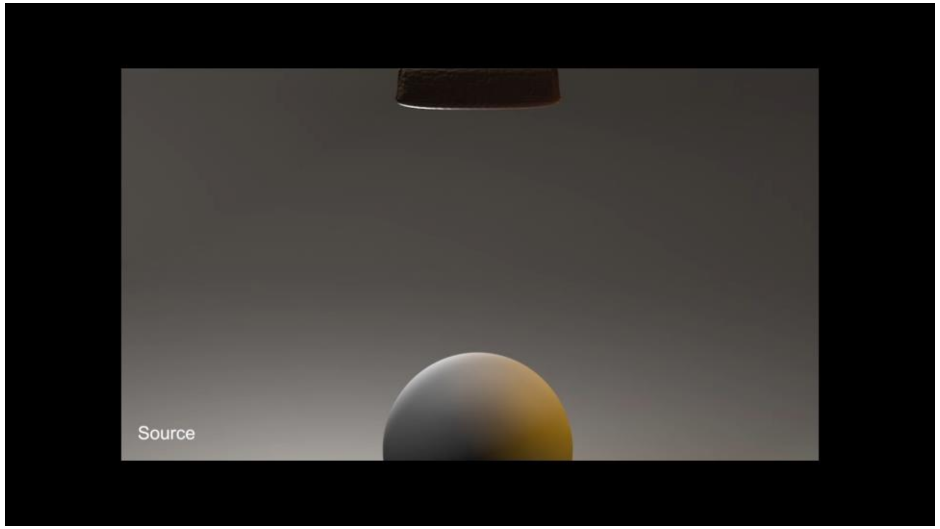


Bender et al. 2017

Weiler et al. 2018



Peer et al. 2018



Kim et al. 2020

Overview

Introduction

Foundations of SPH and PDE Discretization

Material Models

Incompressibility and Pressure Solvers

Data-Driven Approaches and ML Models

Conclusion

Myths about SPH

"Grid approaches are better than SPH"

"An SPH particle represents a molecule"

"SPH is (only) 0^{th}-order consistent"

"SPH suffers from energy loss due to the smoothing"

"An SPH particle represents a droplet"

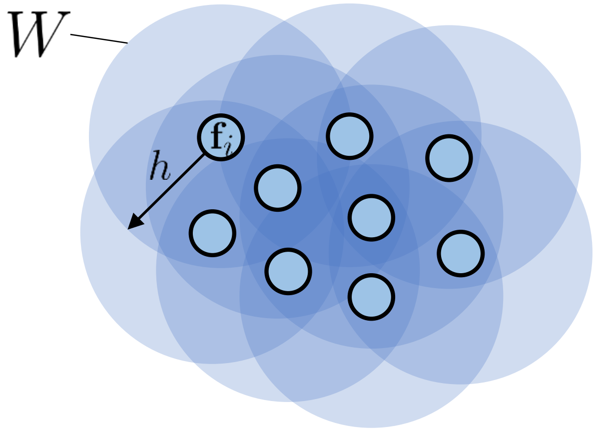
"SPH is better then grid approaches"

"SPH can only be used for fluid dynamics (it's in the name!)"

What is SPH?

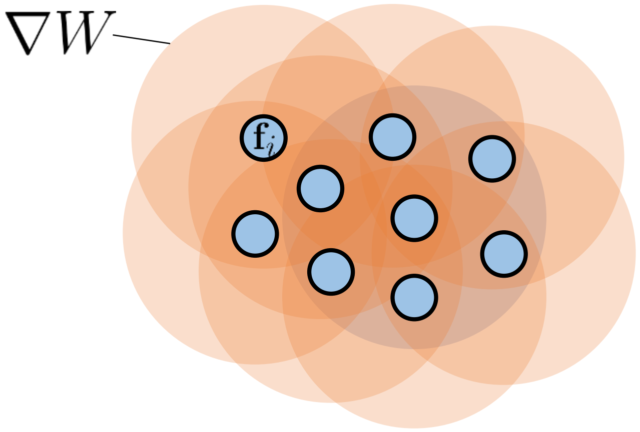
"A mesh-free method for the discretization of spatial field quantities and spatial differential operators"

$$\mathbf{f}(\mathbf{x})$$



$$\sum_i V_i \mathbf{f}_i W(\|\mathbf{x} - \mathbf{x}_i\|, h)$$

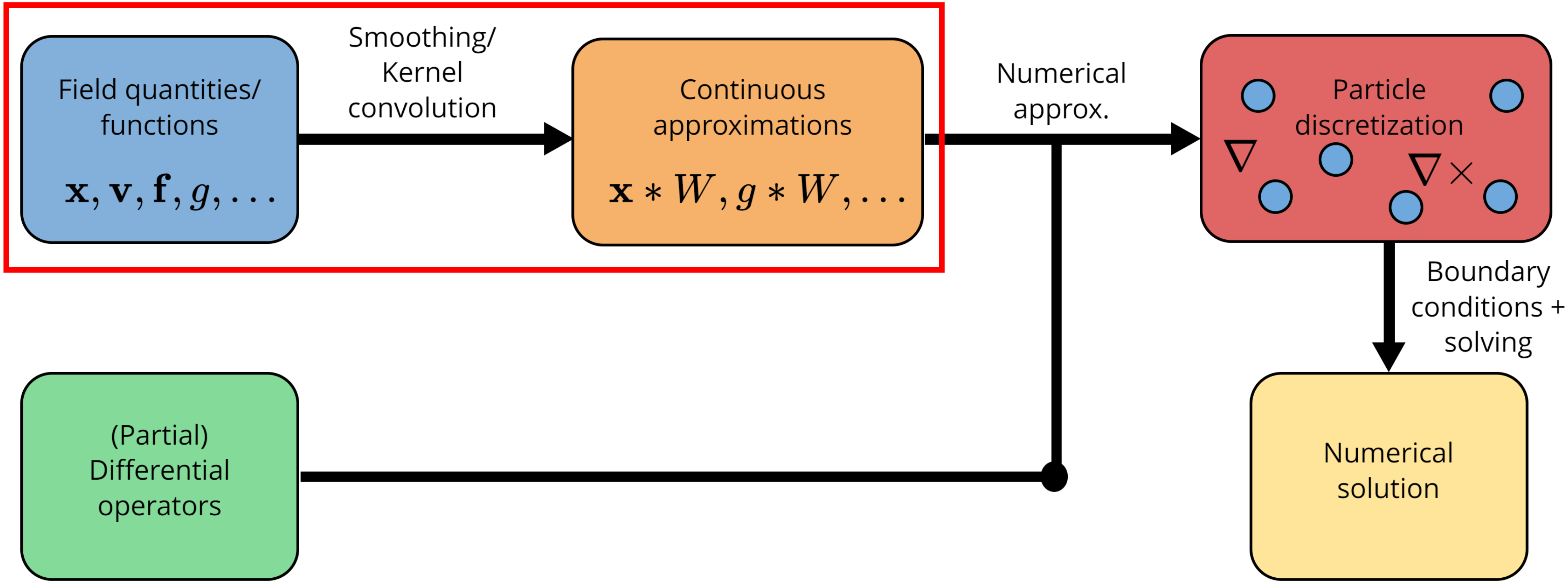
$$\nabla \mathbf{f}$$



$$\sum_i V_i \mathbf{f}_i \nabla W(\|\mathbf{x} - \mathbf{x}_i\|, h)$$

SPH Discretization Pipeline

PDE: $\nabla \times \mathbf{f} + \nabla g = \mathbf{h}(f, g)$



Continuous Approximation

- **The dirac-δ identity** $\delta(\mathbf{r}) = \begin{cases} \infty & \text{if } \mathbf{r} = 0 \\ 0 & \text{otherwise} \end{cases}$, $\int_{\mathbb{R}^d} \delta(\mathbf{r}) dv = 1$

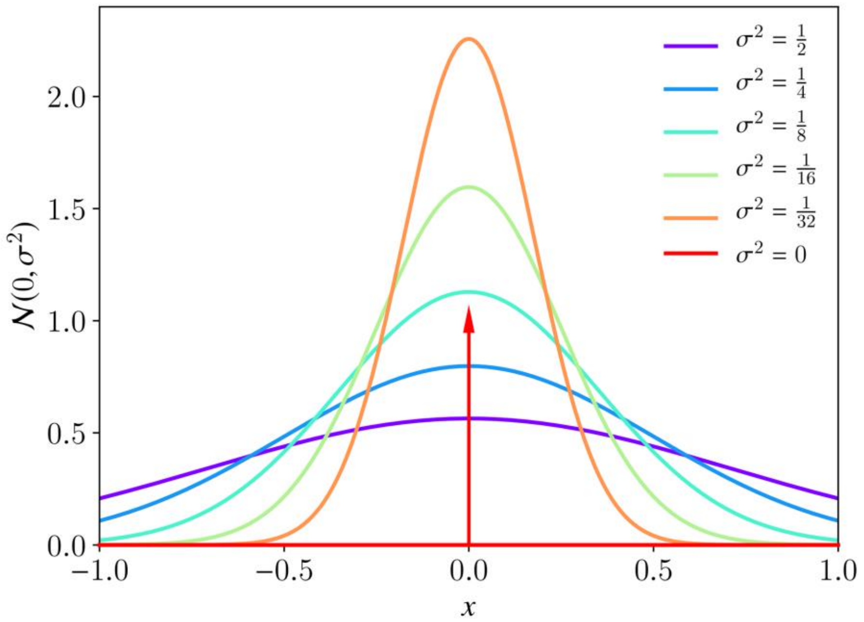
$$A(\mathbf{x}) = (A * \delta)(\mathbf{x}) = \int_{\mathbb{R}^d} A(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')dv$$

- **Approximation with Gaussian kernel** $\mathcal{N}(\mathbf{x}; 0, \sigma^2)$

- **Gaussian zero-variance limit** $\lim_{\sigma \to 0} \mathcal{N}(\mathbf{x}; 0, \sigma^2) = \delta(\mathbf{x})$

- **Replace Gaussian with kernel** $W(\mathbf{x}, h) \approx \mathcal{N}(\mathbf{x}; 0, h)$

- **Why is the convolution necessary?** $\nabla A? \xrightarrow{\text{(shift onto kernel)}} \int_{\mathbb{R}^d} A(\mathbf{x}')\nabla W(\mathbf{x} - \mathbf{x}', h)d\mathbf{x}'$



Continuous Approximation – Kernel Choice

Essential Properties

$$\lim_{h' \to 0} W(\mathbf{r}, h') = \delta(\mathbf{r})$$ (Dirac delta condition)

$$\int_{\mathbb{R}^d} W(\mathbf{r}', h) = 1$$ (normalization condition)

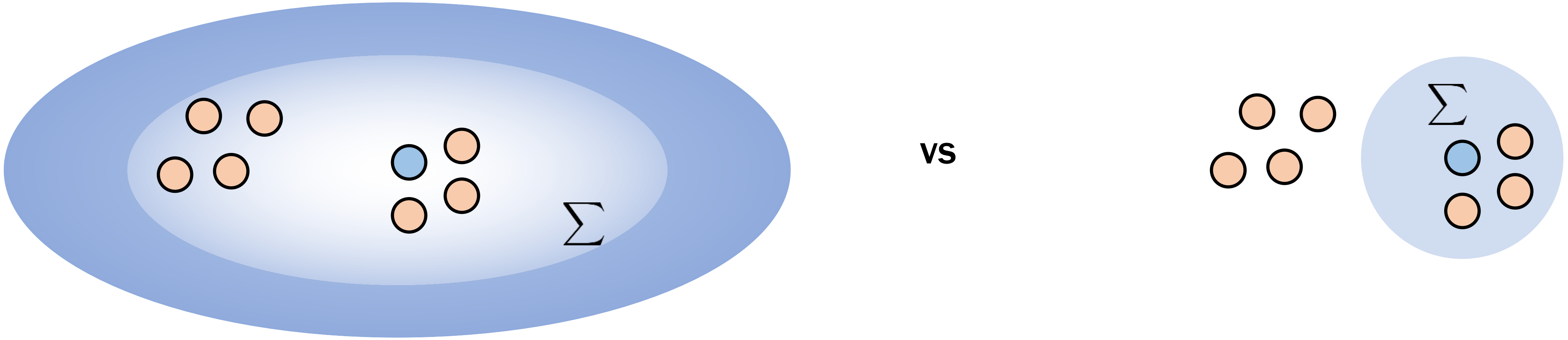
Useful Properties (optional)

$$W(\mathbf{r}, h) \geq 0$$ (positivity condition)

$$W(\mathbf{r}, h) = W(-\mathbf{r}, h)$$ (symmetry condition)

$$W(\mathbf{r}, h) = 0 \text{ for } \|\mathbf{r}\| \geq \bar{h}$$ (compact support cond.)

Global vs Compact Support



Continuous Approximation – Consistency

- How accurate is the continuous approximation?

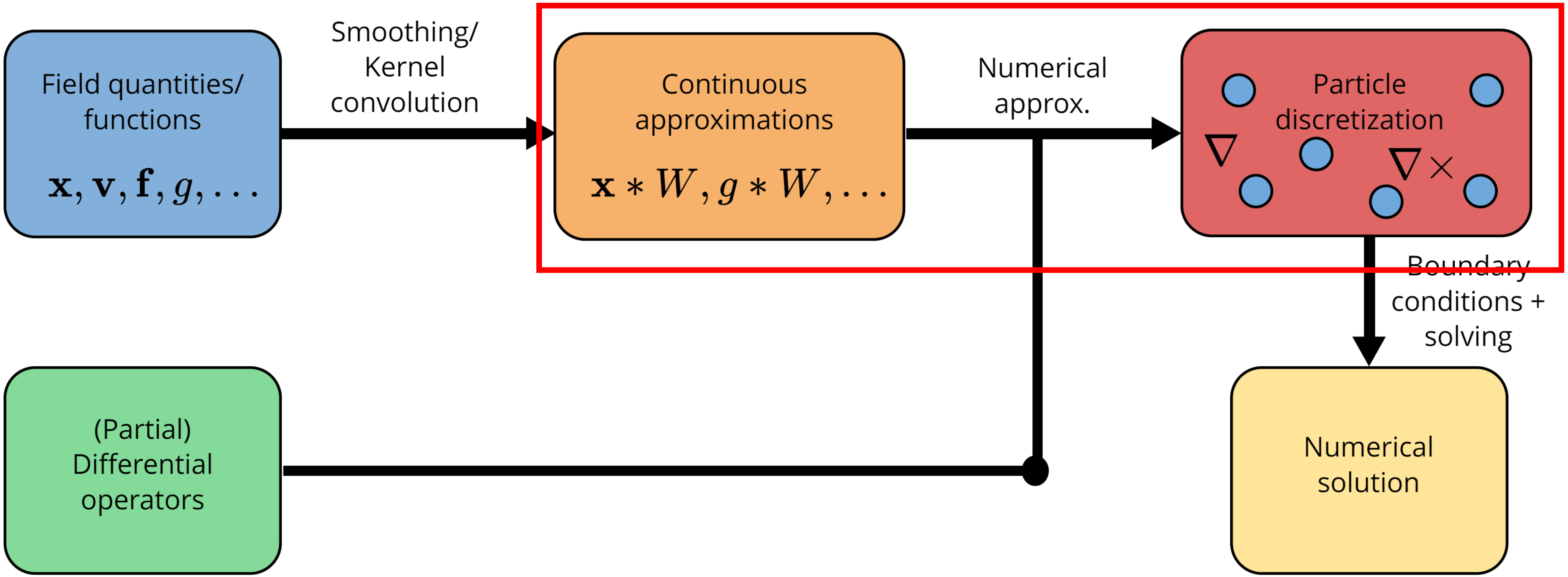
$$(A * W)(\mathbf{x}) \;=\; A(\mathbf{x}) \underbrace{\int W(\mathbf{x} - \mathbf{x}', h) dv'}_{\substack{=1 \\ \text{if kernel normalized}}} + \nabla A|_{\mathbf{x}} \cdot \underbrace{\int (\mathbf{x} - \mathbf{x}') W(\mathbf{x} - \mathbf{x}', h) dv'}_{\substack{=0 \\ \text{if kernel symmetric}}} + \mathcal{O}(\|\mathbf{x} - \mathbf{x}'\|^2)$$

- Normalized, symmetric kernel is 1st-order consistent

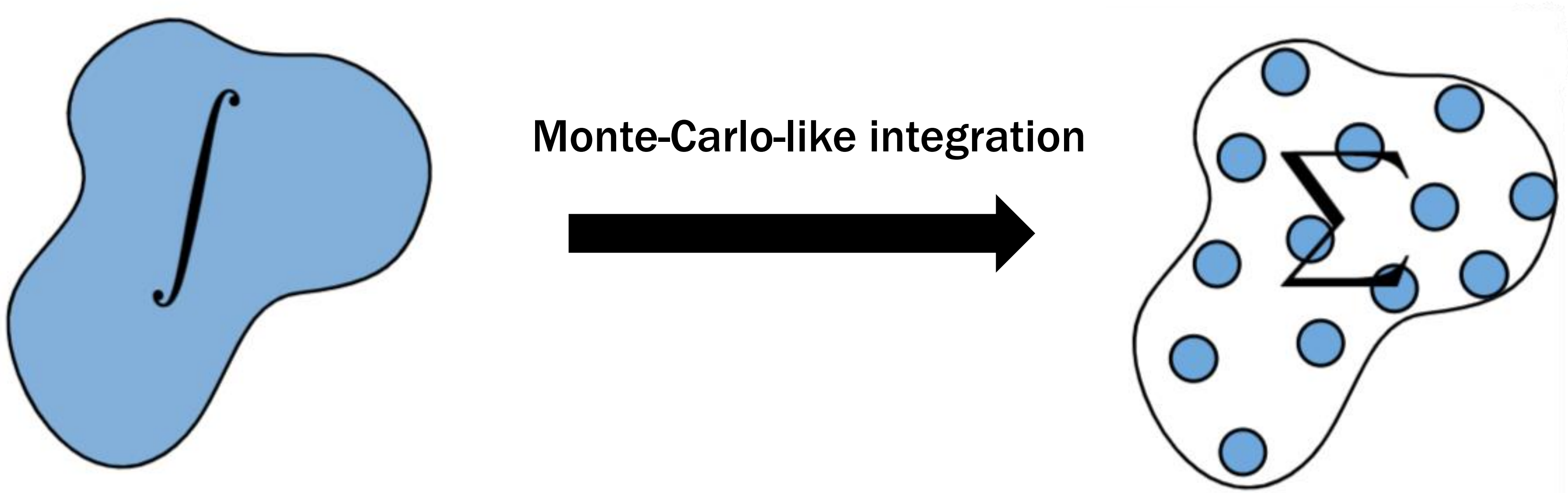
- Specialized kernels with higher-order consistency can be constructed (see [LL10])

SPH Discretization Pipeline

PDE: $\nabla \times \mathbf{f} + \nabla g = \mathbf{h}(f, g)$



Particle Discretization



$$(A * W)(\mathbf{x}) = \int \frac{A(\mathbf{x}')}{\rho(\mathbf{x}')} W(\mathbf{x}-\mathbf{x}', h) \underbrace{\rho(\mathbf{x}') dv'}_{dm'} \approx \sum_i A_i \frac{m_i}{\rho_i} W(\mathbf{x}-\mathbf{x}_i, h) := \langle A \rangle$$

$$\Rightarrow \rho(\mathbf{x}) \approx \sum_j m_j W(\mathbf{x}-\mathbf{x}_j, h)$$

Particle Discretization – Consistency

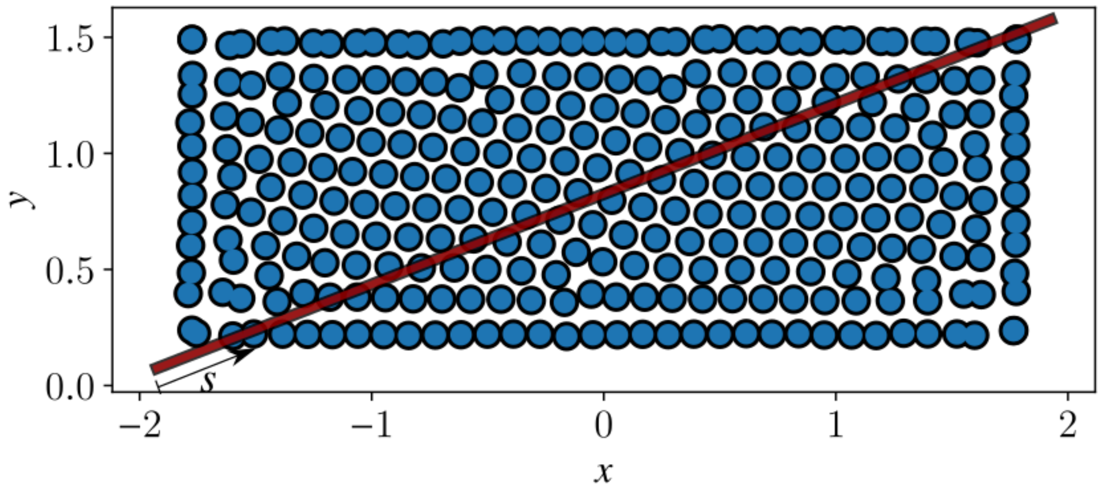
- How accurate is the discrete approximation?

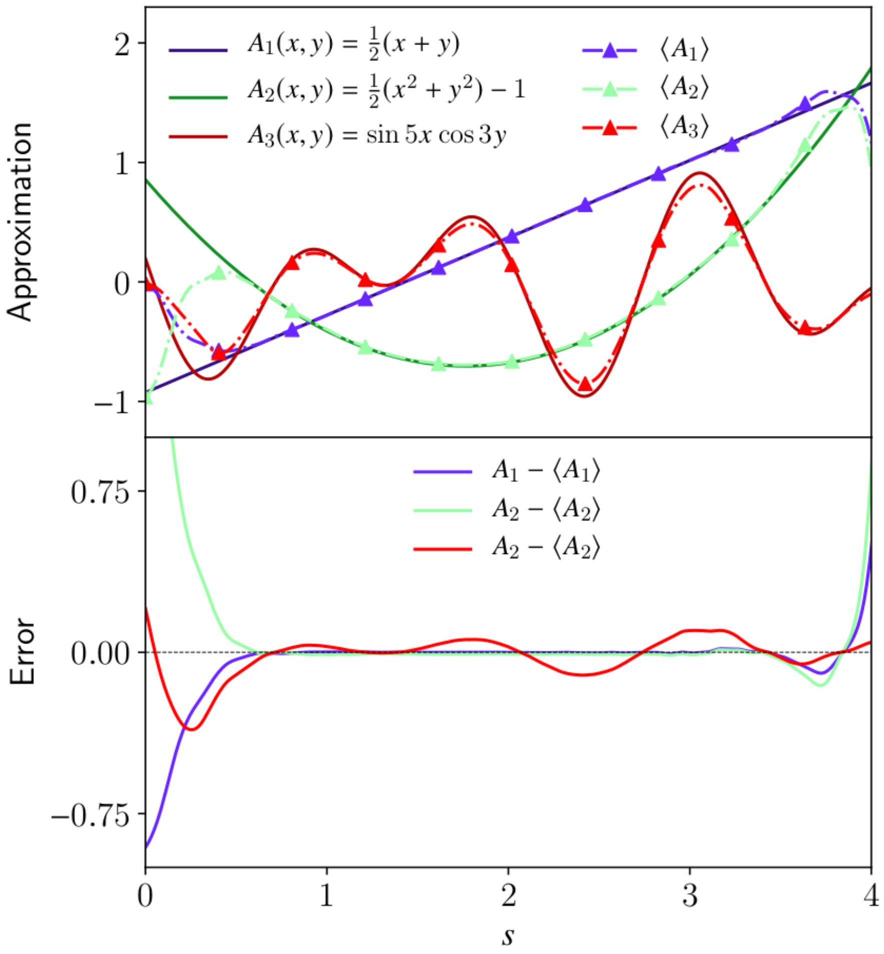
$$A(\mathbf{x}_i) \approx \sum_j A_j \frac{m_j}{\rho_j} W_{ij} = A_i \underbrace{\sum_j \frac{m_j}{\rho_j} W_{ij}}_{\substack{=1 \\ \text{for } 0^{\text{th}}\text{-order} \\ \text{consistency}}} + \nabla A|_{\mathbf{x}_i} \cdot \underbrace{\sum_j \frac{m_j}{\rho_j} (\mathbf{x}_j - \mathbf{x}_i) W_{ij}}_{\substack{=0 \\ \text{for } 1^{\text{st}}\text{-order} \\ \text{consistency}}} + \mathcal{O}(\|\mathbf{x}_j - \mathbf{x}_i\|^2)$$

- Highly dependent on particle ordering
- Is this a problem?
  - Not really! Approximation quality usually still sufficient
  - Alternatively: order recovery
- Also: polynomial error is only half the truth
  - Does not take important physical conservational properties into account

Particle Discretization – Example

- Setting:
  - Rectangular domain discretized into particles
  - Test function sampled on particles
  - Discretization quality measured along red line
- Results good despite lack of consistency order





Operator Discretization

- Direct discretization
  - Operator “shifts” to kernel function
  - Simple
  - Gradient can be reused in implementations

- Direct form might lead to unstable simulations!

- Improved variants:
  - Difference formula
  - Symmetric formula
  - ...

$$\nabla A_i \approx \sum_j A_j \frac{m_j}{\rho_j} \nabla W_{ij}$$

$$\nabla \mathbf{A}_i \approx \sum_j \mathbf{A}_j \frac{m_j}{\rho_j} \otimes \nabla W_{ij}$$

$$\nabla \cdot \mathbf{A}_i \approx \sum_j \mathbf{A}_j \frac{m_j}{\rho_j} \cdot \nabla W_{ij}$$

$$\nabla \times \mathbf{A}_i \approx -\sum_j \mathbf{A}_j \frac{m_j}{\rho_j} \times \nabla W_{ij}$$

Symmetric Formula

- Derived from discrete Lagrangian in hydrodynamic systems:

$$\nabla A_i \approx \rho_i \left(\frac{A_i}{\rho_i^2} \langle \nabla \rho \rangle + \langle \nabla \left(\frac{A_i}{\rho_i} \right) \rangle \right) = \rho_i \sum_j m_j \left(\frac{A_i}{\rho_i^2} + \frac{A_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

- Approx. not even 0^{th}-order consistent. **BUT**:
  - Momentum conserving
  - Error is guided by particle ordering (non-linear properties, symmetries, conservation, etc.)
  - => Leads to more stable simulations (at least in the momentum-conservation setting)
- Consequence:

Polynomial error analysis is only half the truth

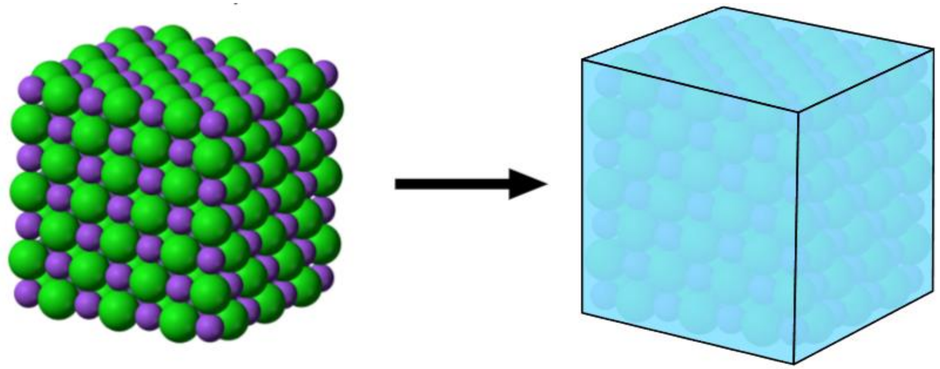
- More information: [Price2012], Smoothed Particle Hydrodynamics and Magnetohydrodynamics
  - Sec. 5 “Why a bad derivative leads to good derivatives: The importance of local conservation”

Continuum Mechanical Models

- How to model fluids and solids?

- Graphics applications:
  - Visual appearance
  - Macroscopic behavior

- What is Continuum Mechanics?
  - Continuously distributed mass
  - Object can be infinitely often divided
  - Models physical phenomena as PDE, e.g.,
    - Linear or (hyper-)elasticity
    - Navier-Stokes eq.
    - Continuum eq.



$$\rho \frac{D^2 \mathbf{x}}{Dt^2} = \nabla \cdot \mathbf{T} + \mathbf{f}_{\mathrm{ext}}$$

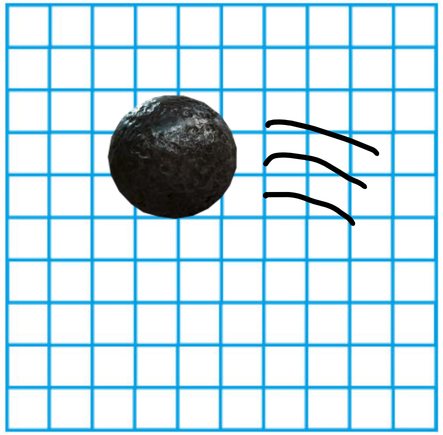
- Linear momentum conservation
- $\frac{D(\cdot)}{Dt}$ denotes the material derivative
- Functions modelling $\mathbf{T}$ are called constitutive relations

Lagrangian vs. Eulerian Point-of-View

Lagrangian Coordinates

- Identify or label material of fluid/solid
- Track material particles as they move
- Monitor changes in its properties
- Field: $A_p^L(t)$

$$\frac{DA^L}{Dt} = \frac{\partial A^L}{\partial t}$$



Eulerian Coordinates

- Identify (or label) fixed locations
- Observe fixed location (like a sensor)
- Monitor changes in properties at these
- Field: $A^E(t, \mathbf{x})$

$$\frac{DA^E}{Dt} = \frac{\partial A^E}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} A^E$$