Smoothed Particle Hydrodynamics

Techniques for the Physics Based Simulation of Fluids and Solids

Elastic Solids

Dan Koschier

Jan Bender

Barbara Solenthaler

Matthias Teschner

UCL

RWTH

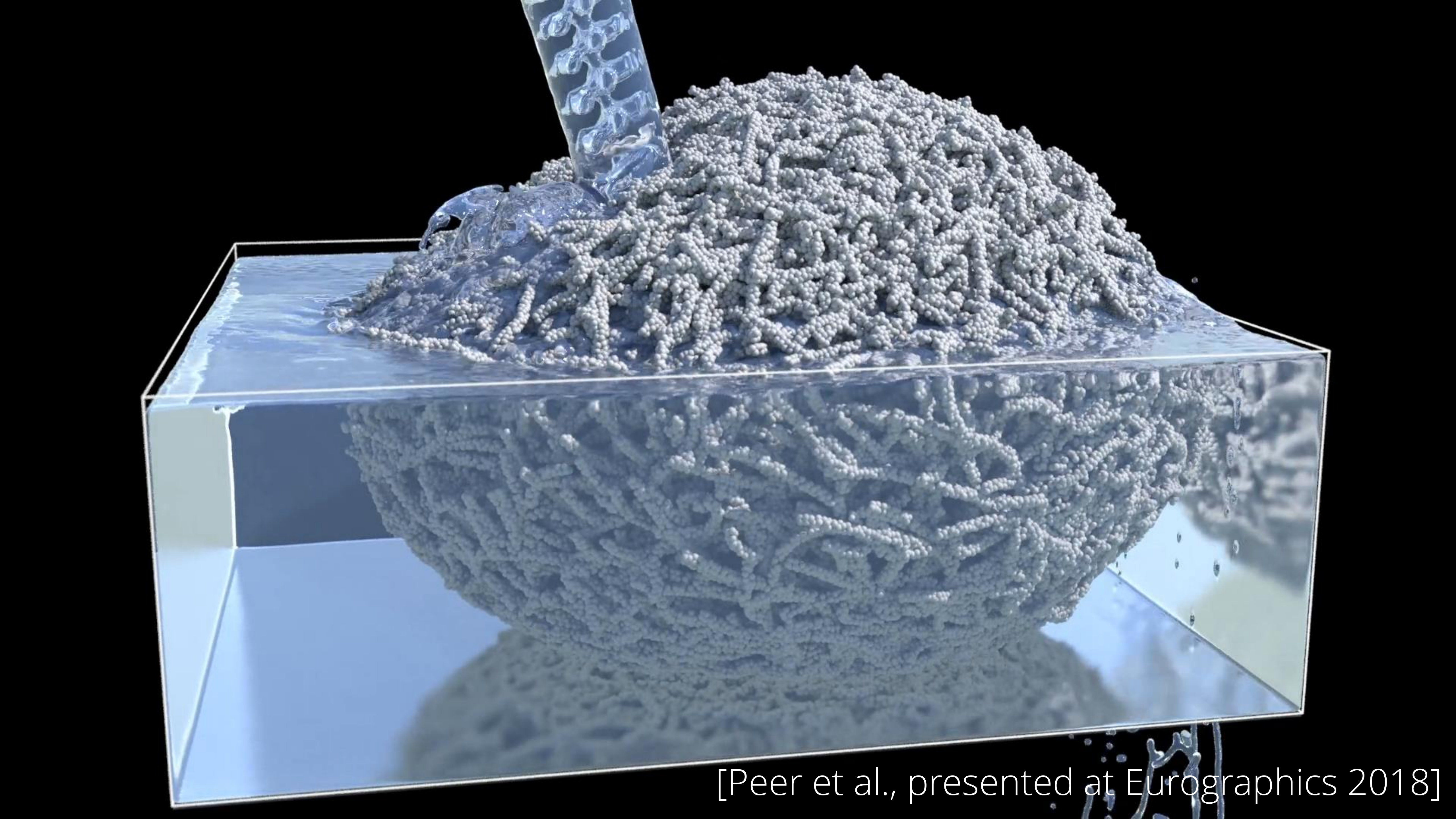
ETH zürich

UNI FREIBURG

Graphics Research - SPH Solver

- Fluids
  - Low viscosity [Mueller 2003, Bender 2017]
  - High viscosity [Debrun 1996, Peer 15, Takahashi 15, Weiler 18]
  - Ferrofluids [Huang 2019]
- Granular materials
- Elastic solids [Solenthaler 2007, Peer 2018]
- Plastic solids [Gerszewski 2009]
- Rigid bodies [Gissler 2019]

[Peer et al., presented at Eurographics 2018]



[Peer et al., presented at Eurographics 2018]

Outline

- Elastic force model
- SPH implementation
- Implicit formulation

Linear Elasticity

– Continuum mechanics formulation

Goal:	Computation of forces from deformations, e.g. shear, compression, stretch
Input:	Current deformed object state, initial undeformed state
Linearity:	Forces depend linearly on object positions

Linear Elasticity – Force Computation

- Initial position $\boldsymbol{x}^0$
- Current position $\boldsymbol{x}$
- Displacement $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}^0$
- Deformation map $\boldsymbol{\Phi}(\boldsymbol{x}^0) = \boldsymbol{x}^0 + \boldsymbol{u} = \boldsymbol{x}$
- Deformation gradient $\boldsymbol{J}(\boldsymbol{\Phi}) = \nabla^0 \boldsymbol{\Phi} = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}^0} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}^0} = \mathbb{1} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}^0}$
- Strain $\boldsymbol{\varepsilon}(\boldsymbol{J}) = \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}^\mathsf{T}) - \mathbb{1}$
- Stress $\boldsymbol{P}(\boldsymbol{\varepsilon}) = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbb{1}$
- Force per volume $\boldsymbol{f}(\boldsymbol{P}) = \nabla \cdot \boldsymbol{P}$

Linear Elasticity - Examples

- Object translation by $\boldsymbol{t}$

$$\boldsymbol{\Phi}(\boldsymbol{x}^0) = \boldsymbol{x}^0 + \boldsymbol{t} = \boldsymbol{x} \qquad \boldsymbol{J}(\boldsymbol{\Phi}) = \mathbb{1} \qquad \boldsymbol{\varepsilon}(\boldsymbol{J}) = \boldsymbol{0}$$

- Isotropic compression or stretch with γ

$$\boldsymbol{\Phi}(\boldsymbol{x}^0) = \gamma \boldsymbol{x}^0 \qquad \boldsymbol{J}(\boldsymbol{\Phi}) = \gamma \mathbb{1} \qquad \boldsymbol{\varepsilon}(\boldsymbol{J}) = (\gamma - 1)\mathbb{1}$$

- Rotation with $\boldsymbol{R}$

$$\boldsymbol{\Phi}(\boldsymbol{x}^0) = \boldsymbol{R}\boldsymbol{x}^0 \qquad \boldsymbol{J}(\boldsymbol{\Phi}) = \boldsymbol{R} \qquad \boldsymbol{\varepsilon}(\boldsymbol{J}) = \frac{1}{2}(\boldsymbol{R} + \boldsymbol{R}^\mathsf{T}) - \mathbb{1}$$

Linear Elasticity - Examples

- Stress $\boldsymbol{P}(\boldsymbol{\varepsilon}) = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbb{1}$ scales strain
  - Strain components are scaled differently with $\mu > 0, \lambda > 0$
- Body force $\boldsymbol{f}(\boldsymbol{P}) = \nabla \cdot \boldsymbol{P}$ accelerates material from high to low deformation
- Isotropic compression / stretch

  $\boldsymbol{\varepsilon}(\boldsymbol{J}) = (\gamma - 1)\mathbb{1}$

  $\boldsymbol{P}(\boldsymbol{\varepsilon}) = (2\mu + 3\lambda)(\gamma - 1)\mathbb{1}$

  $\boldsymbol{f}(\boldsymbol{P}) = \nabla \cdot (2\mu + 3\lambda)(\gamma - 1)\mathbb{1} = -\nabla(2\mu + 3\lambda)(1 - \gamma)$

Linear Elasticity - Discussion

- Enables stable, efficient, simple implicit formulations
- Cannot handle rotating objects
- Cannot handle “large” deformations
  - Limited to incompressible elastic solids

Outline

- Elastic force model
- SPH implementation
- Implicit formulation

Kernel Gradient Correction

– Deformation gradient

$$\boldsymbol{J}(\boldsymbol{x}, \boldsymbol{x}^0) = \mathbb{1} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}^0} \qquad \boldsymbol{J}_i = \mathbb{1} + \sum_j^0 V_j^0 (\boldsymbol{x}_{ji} - \boldsymbol{x}_{ji}^0) \otimes \nabla W(\boldsymbol{x}_{ij}^0)$$

– SPH kernel gradient correction,
e.g. [Bonet and Lok 1999]

$$\nabla \tilde{W}_i(\boldsymbol{x}_{ij}^0) = \left(\sum_j^0 V_j^0 \nabla W(\boldsymbol{x}_{ij}^0) \otimes \boldsymbol{x}_{ji}^0 \right)^{-1} \nabla W(\boldsymbol{x}_{ij}^0)$$

– First-order consistent deformation gradient

$$\boldsymbol{J}_i = \mathbb{1} + \sum_j^0 V_j^0 (\boldsymbol{x}_{ji} - \boldsymbol{x}_{ji}^0) \otimes \nabla \tilde{W}(\boldsymbol{x}_{ij}^0)$$

Corotated Formulation

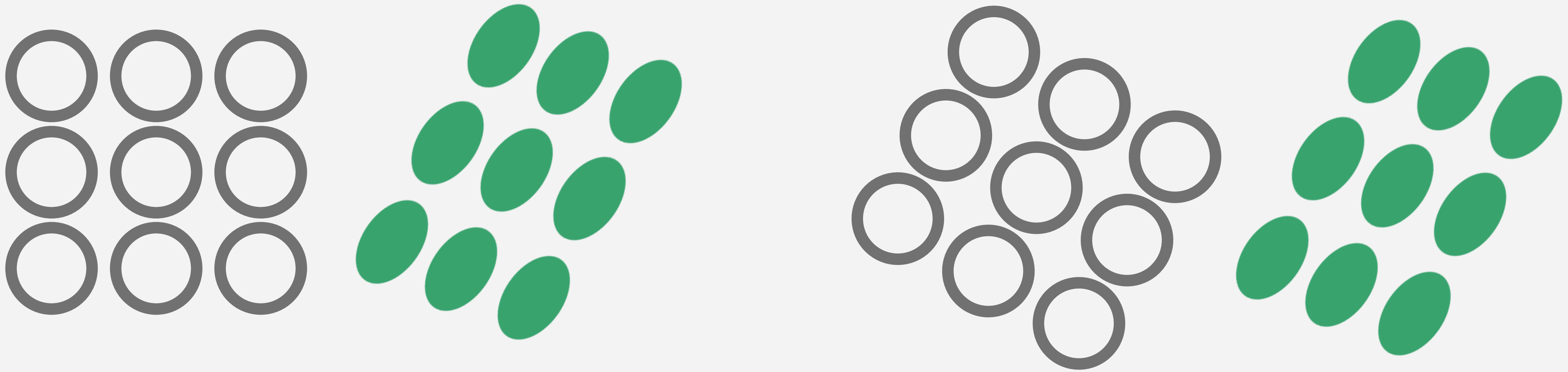
- Linear elasticity misinterprets rotations as deformations
- Rotation $\boldsymbol{R}_i$ is extracted from deformation gradient $\boldsymbol{J}_i$, e.g. polar decomposition or [Mueller 2016]
- Reference configuration is rotated with the object
- Kernel gradient with rotated reference configuration

$$\nabla \overset{\star}{W}_i(\boldsymbol{x}_{ij}^0) = \boldsymbol{R}_i \left(\sum_j^0 V_j^0 \nabla W(\boldsymbol{x}_{ij}^0) \otimes \boldsymbol{x}_{ij}^0 \right)^{-1} \nabla W(\boldsymbol{x}_{ij}^0)$$

- Deformation map with rotated reference configuration

$$\overset{\star}{\boldsymbol{J}}_i = \mathbb{1} + \sum_j^0 V_j^0 (\boldsymbol{x}_{ji} - \boldsymbol{R}_i \boldsymbol{x}_{ji}^0) \otimes \nabla \overset{\star}{W}_i(\boldsymbol{x}_{ij}^0)$$

Corotated Formulation - Illustration



$\boldsymbol{x}^0$ $\boldsymbol{x}$ $\boldsymbol{R}\boldsymbol{x}^0$ $\boldsymbol{x}$

$$\boldsymbol{J}(\boldsymbol{x}^0) = \boldsymbol{S}\boldsymbol{R} \qquad \boldsymbol{J}(\boldsymbol{R}\boldsymbol{x}^0) = \boldsymbol{S}$$

SPH Force Computation

- Strain $\boldsymbol{\varepsilon}_i = \frac{1}{2}(\overset{\star}{\boldsymbol{J}}_i + \overset{\star}{\boldsymbol{J}}_i^{\mathsf{T}}) - \mathbb{1}$
- Stress $\boldsymbol{P}_i = 2\mu\boldsymbol{\varepsilon}_i + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_i)\mathbb{1}$
- Stress with alternative parameter formulation
  - Separation of shear and bulk stress

$$\boldsymbol{P}_i = 2G(\boldsymbol{\varepsilon}_i - \frac{1}{3}\operatorname{tr}(\varepsilon_i))\mathbb{1} + K\operatorname{tr}(\boldsymbol{\varepsilon}_i)\mathbb{1} \qquad G = \mu \qquad K = \lambda + \frac{2\mu}{3}$$

- Force per volume [Ganzenmueller 2015]

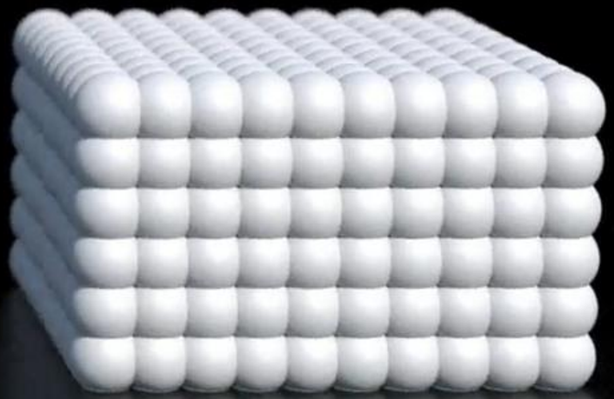
$$\frac{\boldsymbol{F}_i}{V_i} = \sum_j^0 V_j^0 \left(\boldsymbol{P}_i \nabla \overset{\star}{W}_i(\boldsymbol{x}_{ij}^0) - \boldsymbol{P}_j \nabla \overset{\star}{W}_j(\boldsymbol{x}_{ij}^0) \right)$$

Zero-Energy Modes [Ganzenmueller 2015]

- Certain deformations do not result in forces
- Compute correction forces for inconsistent states
- For all vectors $\boldsymbol{x}_{ji}$, the term $\boldsymbol{e}^i_{ji} = \boldsymbol{J}_i \boldsymbol{x}^0_{ji} - \boldsymbol{x}_{ji}$ should be zero
- Penalty force minimizes $\boldsymbol{e}^i_{ji}$

$$\boldsymbol{F}^{ze}_i = -\frac{1}{2}\alpha K V^0_i \sum\nolimits^0_j V^0_j \frac{W(\boldsymbol{R}\boldsymbol{x}^0_{ij})}{\|\boldsymbol{R}\boldsymbol{x}^0_{ij}\|^2} \left(\frac{\boldsymbol{e}^i_{ji} \cdot \boldsymbol{x}_{ji}}{\|\boldsymbol{x}_{ji}\|} + \frac{\boldsymbol{e}^i_{ij} \cdot \boldsymbol{x}_{ij}}{\|\boldsymbol{x}_{ij}\|} \right) \frac{\boldsymbol{x}_{ji}}{\|\boldsymbol{x}_{ji}\|}$$

standard SPH



with kernel correction

Outline

- Elastic force model
- SPH implementation
- Implicit formulation

Linear System

- Explicit: $\boldsymbol{v}^{t+\Delta t} = \boldsymbol{v}^t + \Delta t \frac{\boldsymbol{F}(\boldsymbol{x}^t, \boldsymbol{x}^0)}{m}$
- Implicit: $\boldsymbol{v}^{t+\Delta t} = \boldsymbol{v}^t + \Delta t \frac{\boldsymbol{F}(\boldsymbol{x}^{t+\Delta t}, \boldsymbol{x}^0)}{m}$
- Position update: $\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \boldsymbol{v}^{t+\Delta t}$
- Implicit: $\boldsymbol{v}^{t+\Delta t} = \boldsymbol{v}^t + \Delta t \frac{\boldsymbol{F}(\boldsymbol{x}^t + \Delta t \boldsymbol{v}^{t+\Delta t}, \boldsymbol{x}^0)}{m}$
- Linear force formulation:

$$\boldsymbol{F}(\boldsymbol{x}^t + \Delta t \boldsymbol{v}^{t+\Delta t}, \boldsymbol{x}^0) = \boldsymbol{F}(\boldsymbol{x}^t, \boldsymbol{x}^0) + \boldsymbol{F}(\Delta t \boldsymbol{v}^{t+\Delta t}, \boldsymbol{0})$$

- Linear system

$$\boldsymbol{v}^{t+\Delta t} - \Delta t \frac{\boldsymbol{F}(\Delta t \boldsymbol{v}^{t+\Delta t}, \boldsymbol{0})}{m} = \boldsymbol{v}^t + \Delta t \frac{\boldsymbol{F}(\boldsymbol{x}^t, \boldsymbol{x}^0)}{m}$$

Solver

- Linear, symmetric system
- Matrix-free solver implementation
  - See PPE solver
- Iterative solvers, e.g. CG
- Two iterations over particles and neighbors per iteration

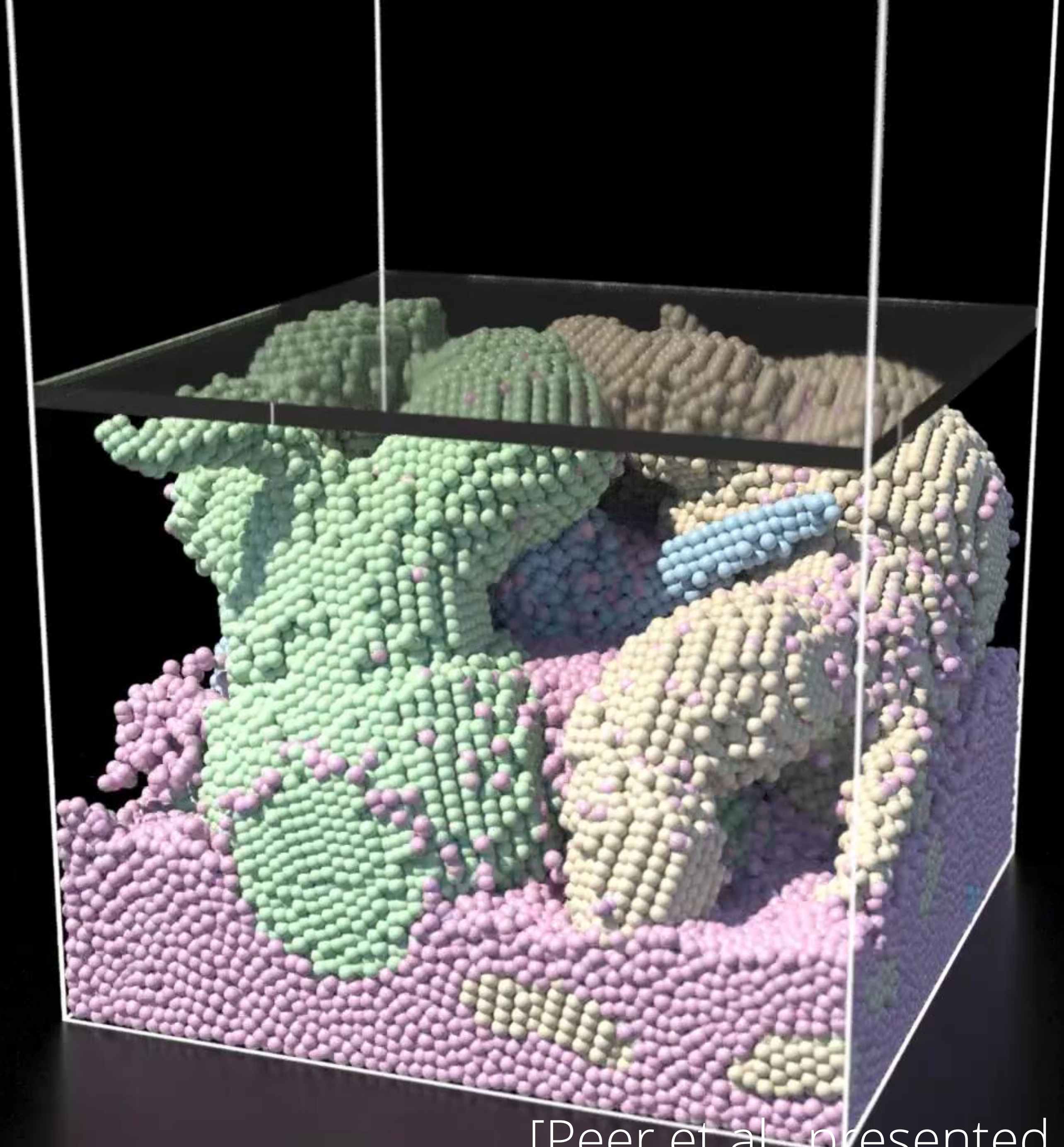
Framework

- Start: $\boldsymbol{x}^t, \boldsymbol{v}^t$
- Gravity, viscosity, ... : $\boldsymbol{v}^*$
- Elastic forces: $\boldsymbol{v}^{**} - \Delta t \frac{\boldsymbol{F}(\Delta t \boldsymbol{v}^{**}, \boldsymbol{0})}{m} = \boldsymbol{v}^* + \Delta t \frac{\boldsymbol{F}(\boldsymbol{x}^t, \boldsymbol{x}^0)}{m}$
- Zero-energy modes: $\boldsymbol{v}^{***} \leftarrow \boldsymbol{v}^{**}$
- Pressure: p
- Velocity update: $\boldsymbol{v}^{t+\Delta t} = \boldsymbol{v}^{***} - \Delta t \frac{1}{\rho} \nabla p$
- Position update: $\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \boldsymbol{v}^{t+\Delta t}$

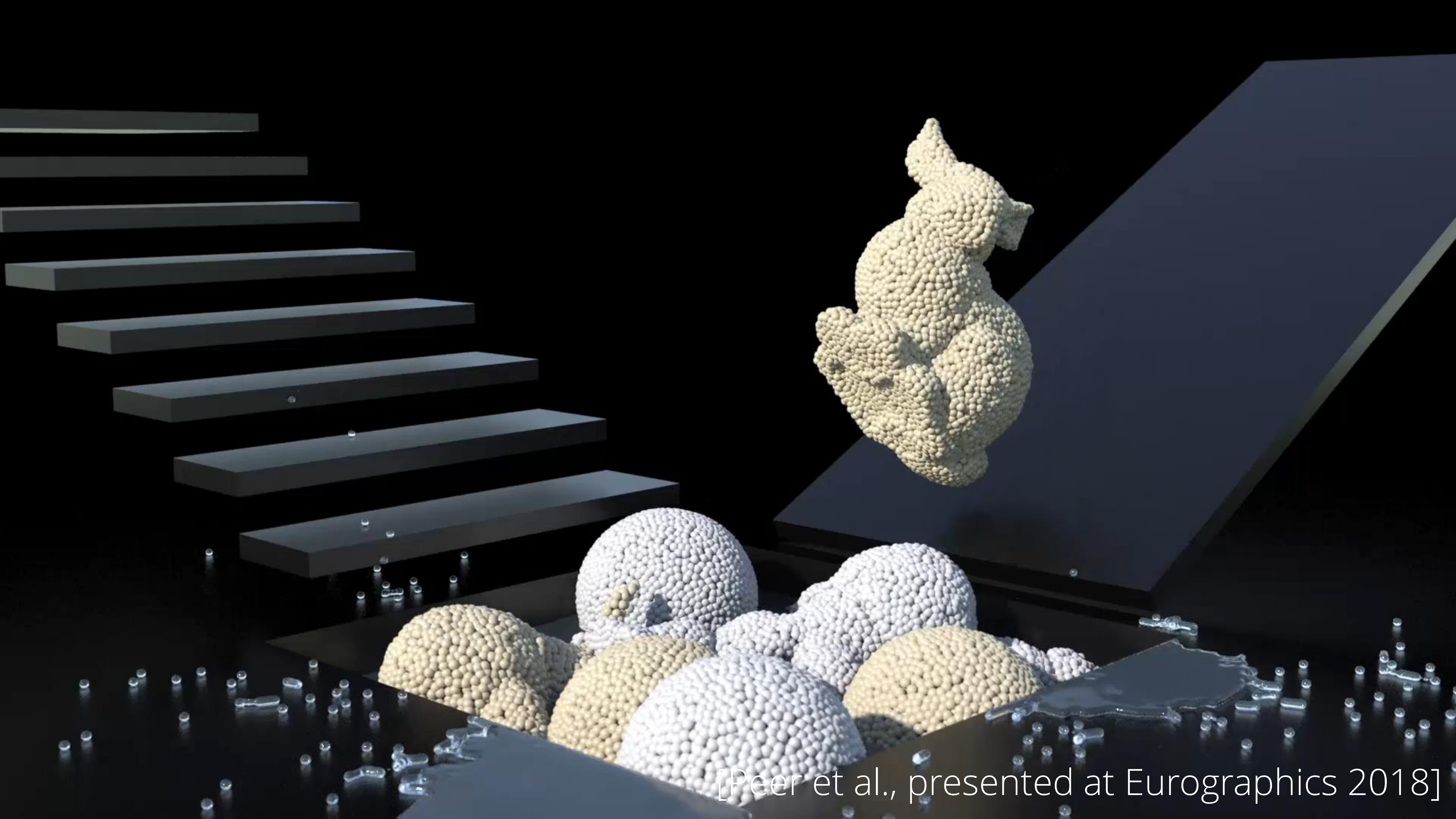
Pressure Projection

- Handles self-collisions, preserves volume
  - Elastic forces preserve volume,
    but do not detect collisions
- Boundary handling with other phases
  - Simply solve pressure for all particles of
    all interacting phases, e.g. elastic solids,
    rigid solids, fluids

[Peer et al., presented at Eurographics 2018]



[Peer et al., presented at Eurographics 2018]



[Peer et al., presented at Eurographics 2018]