Smoothed Particle Hydrodynamics Techniques
for the Physics Based Simulation of Fluids and Solids

VISCOSITY

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Outline

- Motivation
- Viscous Force
- Explicit Viscosity
- Implicit Viscosity
- Results

Motivation

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Classical Newtonian Fluid Model

Linear momentum equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \mathbf{f}$$

- Density ρ
- Velocity $\mathbf{v}$
- Stress tensor $\mathbf{T}$
- Force density $\mathbf{f}$

Newtonian constitutive model:

$$\mathbf{T} = -p\mathbb{1} + 2\mu\mathbf{E}$$

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)$$

- Strain rate tensor $\mathbf{E}$
- Pressure p
- Identity $\mathbb{1}$
- Dynamic viscosity μ

Navier-Stokes equations: $$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}$$

Viscous Force

- Viscous force for incompressible fluids:

$$\mathbf{f}_{\text{visco}} = \mu \nabla^2 \mathbf{v}$$

- Recent SPH solvers either compute the divergence of the strain rate $\mathbf{E}$ or directly determine the Laplacian of $\mathbf{v}$.

- Note that strain rate based approaches must enforce a divergence-free velocity field to avoid undesired bulk viscosity.

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Explicit Viscosity

- Standard SPH discretization of this Laplacian:

$$\nabla^2 \mathbf{v}_i = \sum_j \frac{m_j}{\rho_j} \mathbf{v}_j \nabla^2 W_{ij}$$

- Disadvantages:
  - Sensitive to particle disorder [Mon05, Pri12]
  - The Laplacian of the kernel changes its sign inside the support radius

Explicit Viscosity

- Alternative: determine one derivative using SPH and the second one using finite differences.

$$\nabla^2 v_i \approx 2(d+2) \sum_j \frac{m_j}{\rho_j} \frac{\mathbf{v}_{ij} \cdot \mathbf{x}_{ij}}{\|\mathbf{x}_{ij}\|^2 + 0.01h^2} \nabla W_{ij}$$

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \;\; \mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j, \;\; d = \text{dimension}$$

- Advantages:
  - Galilean invariant
  - vanishes for rigid body rotation
  - conserves linear and angular momentum

XSPH

- Core idea of XSPH: reduce the particle disorder by smoothing the velocity field:

$$\hat{\mathbf{v}}_i = \mathbf{v}_i + \alpha \sum_j \frac{m_j}{\rho_j}(\mathbf{v}_j - \mathbf{v}_i)W_{ij}$$

- XSPH can also be used as artificial viscosity model.
- Advantage: The second derivative is not needed.
- Disadvantage: α is not physically meaningful.

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Takahashi et al. 2015

- Compute the viscous force as divergence of the strain rate $\mathbf{E}$:

$$\mathbf{f}_{\text{visco}} = \mu \nabla \cdot \left(\nabla \mathbf{v}_i + (\nabla \mathbf{v}_i)^T \right) = \sum_j m_j \left(\frac{2\mathbf{E}_i}{\rho_i^2} + \frac{2\mathbf{E}_j}{\rho_j^2} \right) \nabla W_{ij}$$

- Implicit integration scheme

$$\mathbf{v}(t + \Delta t) = \mathbf{v}^* + \frac{\Delta t}{\rho} \mathbf{f}_{\text{visco}}(t + \Delta t)$$

- Solve linear system using the conjugate gradient method (CG).

Peer et al. 2015/2016

- Decompose velocity gradient: $\nabla \mathbf{v} = \mathbf{R} + \mathbf{V} + \mathbf{S}$

- Reduce shear rate by user-defined factor $0 \leq \xi \leq 1$:

$$\nabla \mathbf{v}^{\text{target}} = \mathbf{R} + \mathbf{V} + \xi \mathbf{S}$$

- Reconstruct velocity field by solving linear system with CG:

$$\mathbf{v}_i(t + \Delta t) = \frac{1}{\rho_i} \sum_j m_j \left(\mathbf{v}_j(t + \Delta t) + \frac{\nabla \mathbf{v}_i^{\text{target}} + \nabla \mathbf{v}_j^{\text{target}}}{2} \mathbf{x}_{ij} \right) W_{ij}$$

Bender & Koschier 2017

- Define velocity constraint for each particle with user-defined factor:

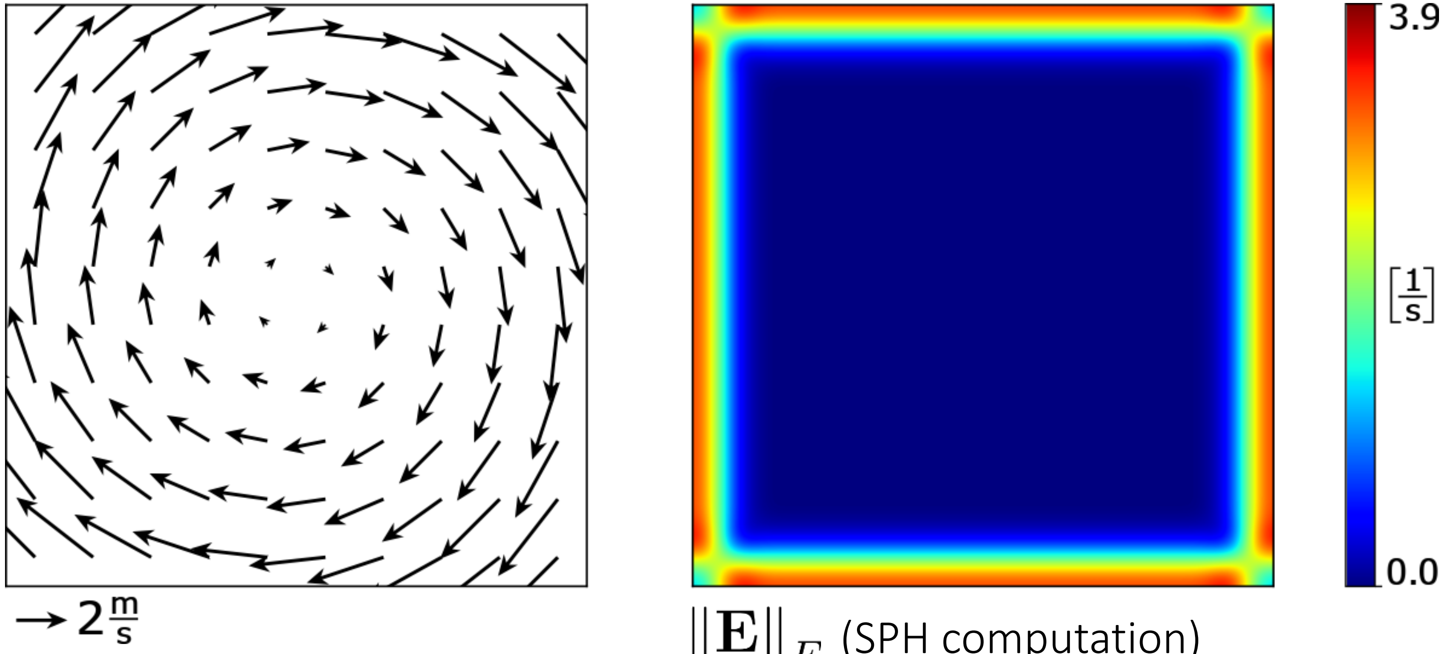
$$\mathbf{C}_i(\mathbf{v}) = \mathbf{E}_i - \gamma \mathbf{E}_i = \mathbf{0}, \quad 0 \leq \gamma \leq 1$$

- The constraint is a 6D function due to the symmetric strain tensor.

- Solve linear system for corresponding Lagrange multipliers.

Strain Rate Computation

- The introduced methods are based on the strain rate.
- However, computing the strain rate using SPH leads to errors at the free surface due to particle deficiency.



$||\mathbf{E}||_F$ (SPH computation)

Weiler et al. 2018

- Implicit integration scheme

$$\mathbf{v}(t+\Delta t) = \mathbf{v}^* + \frac{\Delta t}{\rho}\mu\nabla^2\mathbf{v}(t+\Delta t)$$

- Compute Laplacian as

$$\nabla^2\mathbf{v}_i = 2(d+2)\sum_j \frac{m_j}{\rho_j}\frac{\mathbf{v}_{ij}\cdot\mathbf{x}_{ij}}{\|\mathbf{x}_{ij}\|^2 + 0.01h^2}\nabla W_{ij}$$

- Solve linear system using a meshless conjugate gradient method.
- Laplacian approximation avoids problems at the free surface.

Discussion

- The strain rate based formulation leads to errors and artifacts at the free surface, which is avoided by Weiler et al.
- The viscosity parameters of Bender and Peer depend on the temporal and spatial resolution.
- Peer’s reconstruction of the velocity field is fast but introduces a significant damping => not suitable for low viscous flow
- Takahashi et al. require the second-ring neighbors => low performance

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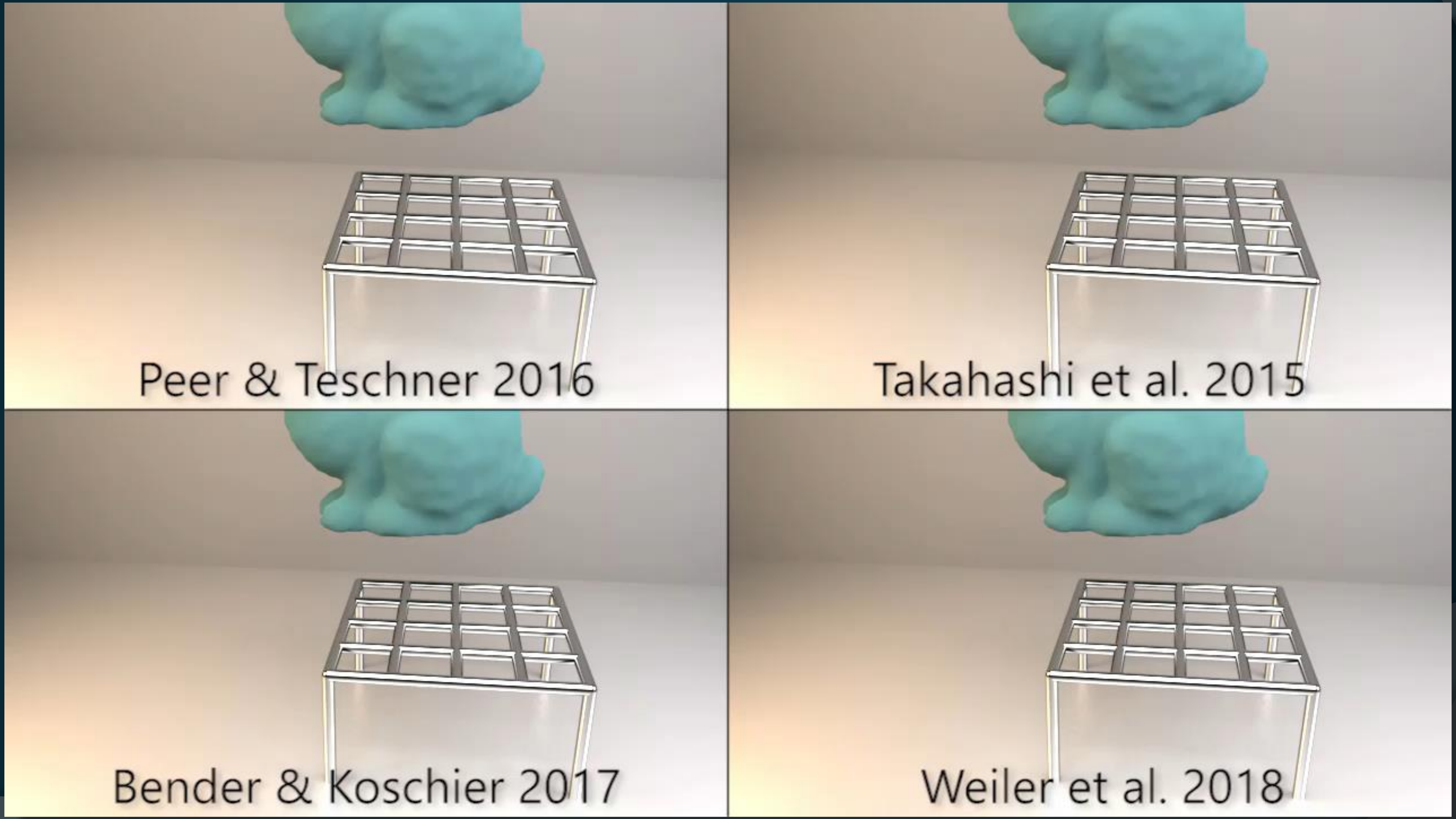
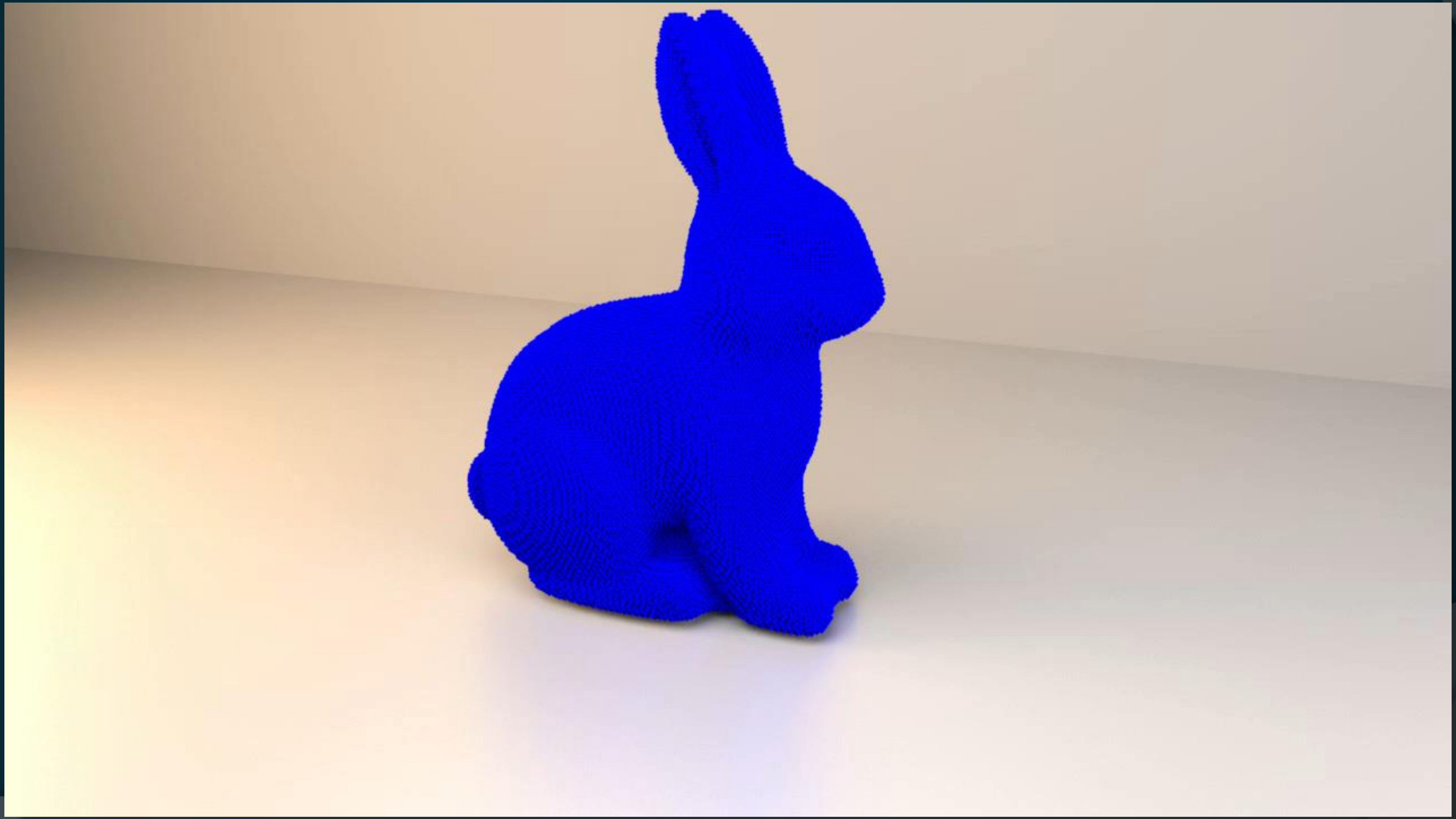
Quality Comparison



Coiling


Peer & Teschner 2016
Bender & Koschier 2017
Takahashi et al. 2015
Weiler et al. 2018

Termperature-Dependent Viscosity



Summary

- Low viscous flow
  - Explicit methods are cheap and well-suited
  - Approximation of Laplacian yields better results while XSPH is slightly faster
- Highly viscous fluids
  - Implicit methods are recommended to guarantee stability
  - Strain rate based SPH formulations lead to artifacts at the free surface
  - Weiler et al. avoid this problem and generate more realistic results.