# Outline

**Block 1 (9:00 - 10:30)**

- Foundations of SPH
- Governing equations
- Time integration
- Example: Our first SPH solver
- Neighborhood Search

Coffee break (30min)

**Block 2 (11:00 - 12:30)**

- Enforcing incompressibility
  - State equation solvers
  - Implicit pressure solvers
- Boundary Handling
  - Particle-based methods
  - Implicit approaches

Lunch break (60min)

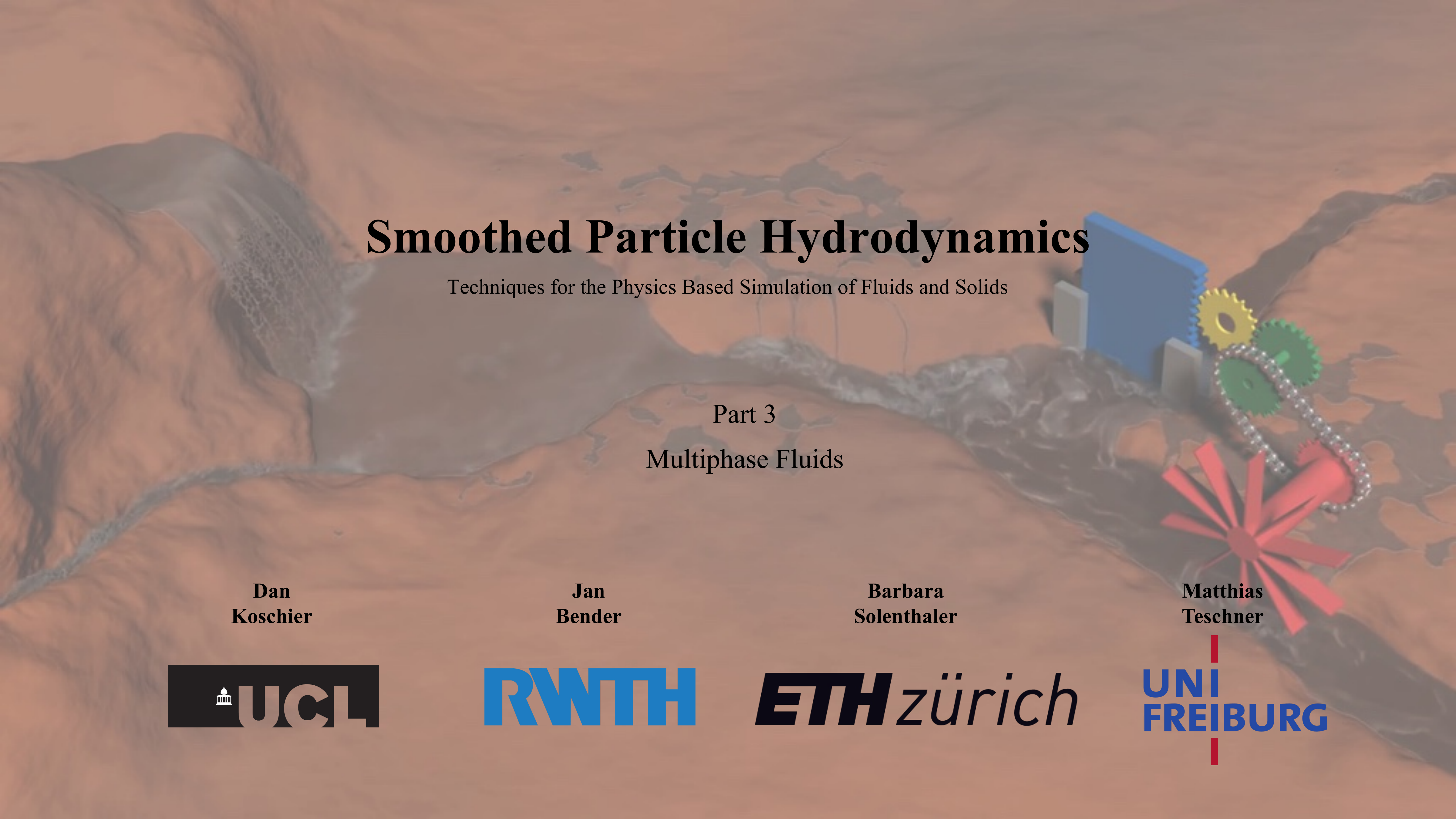
**Block 3 (13:30 - 15:00)**

- Multiphase fluids
- Highly-viscous fluids
- Vorticity and turbulent fluids
- Demo: SPlisHSPlasH

Coffee break (30min)

**Block 4 (15:30 - 17:00)**

- Deformable solids
- Rigid body simulation
  - Dynamics and coupling
- Data-driven/ML techniques
- Summary and conclusion



# Smoothed Particle Hydrodynamics

Techniques for the Physics Based Simulation of Fluids and Solids

## Part 3

## Multiphase Fluids


**Dan Koschier** — UCL

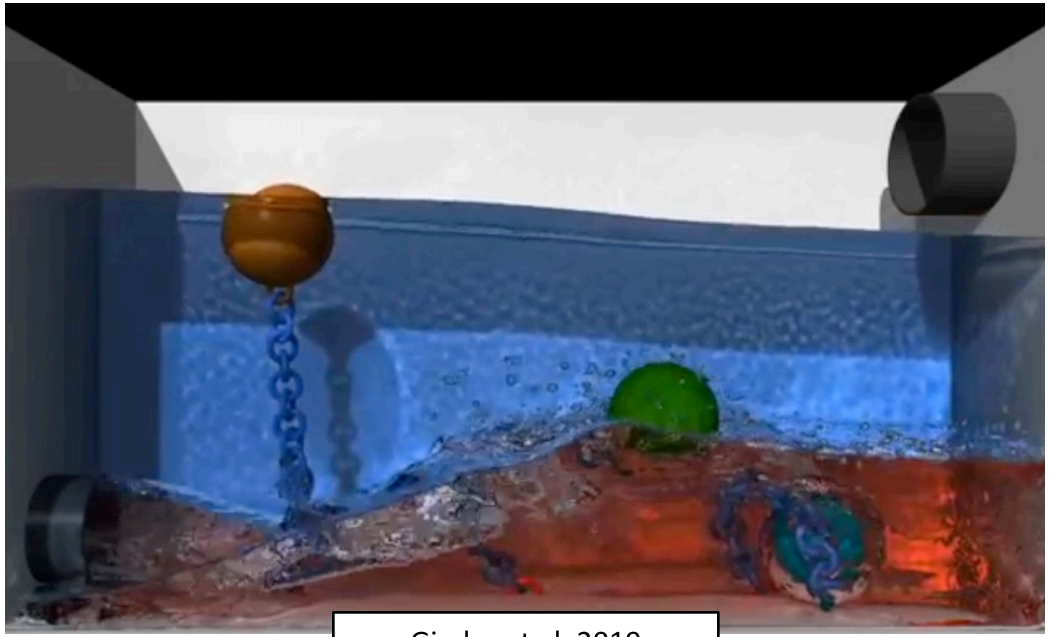
**Jan Bender** — RWTH

**Barbara Solenthaler** — ETH zürich

**Matthias Teschner** — UNI FREIBURG

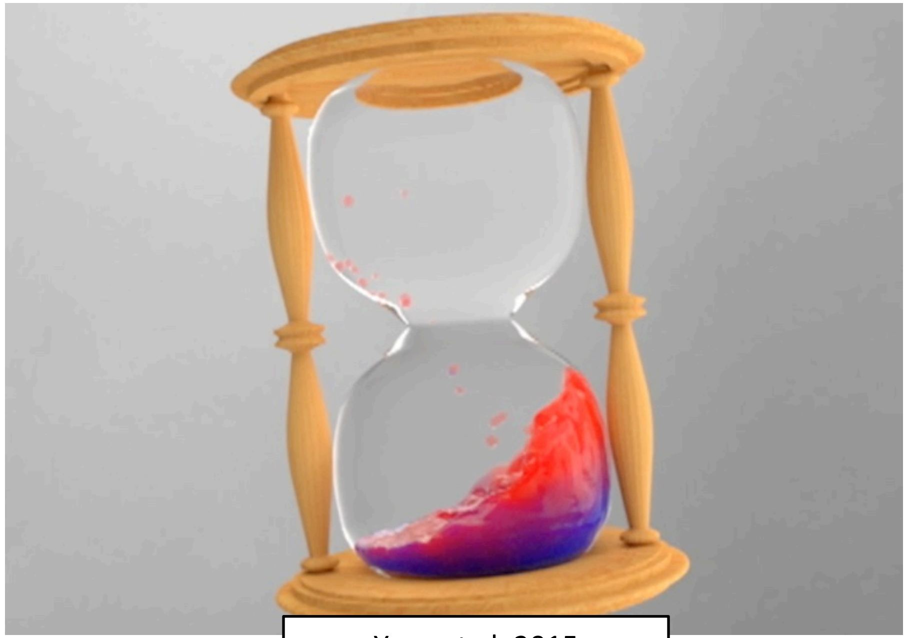
# Motivation

## Fluid Interfaces



Gissler et al. 2019

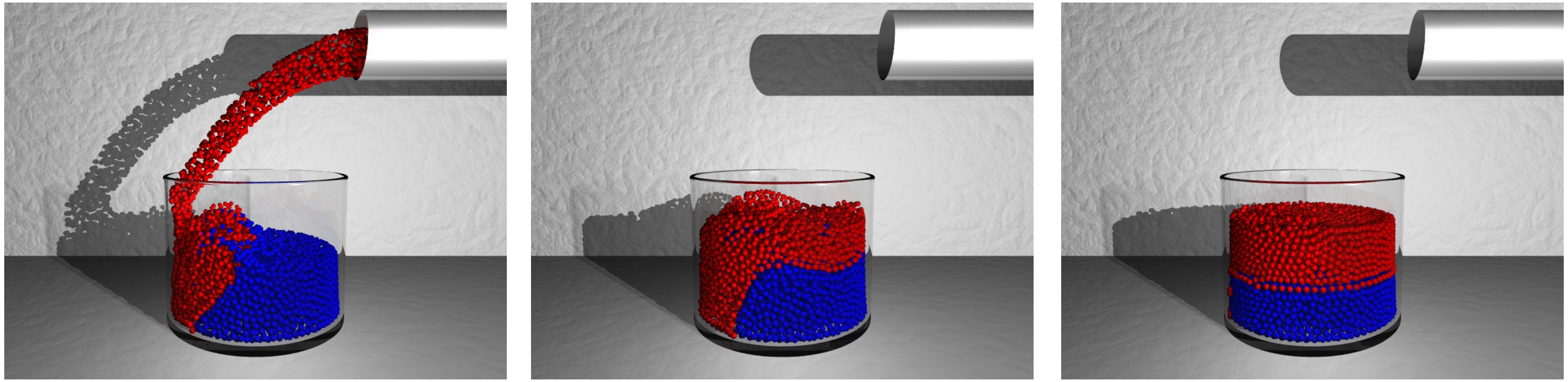
## Complex mixing phenomena



Yang et al. 2015

# My First Multi-fluid SPH Solver

- Particles carry attributes individually
  - Mass, rest density
  - Concentration, temperature, viscosity, ...
- Two fluids a and b, with $\frac{m_a}{\rho_a^0} = \frac{m_b}{\rho_b^0}$



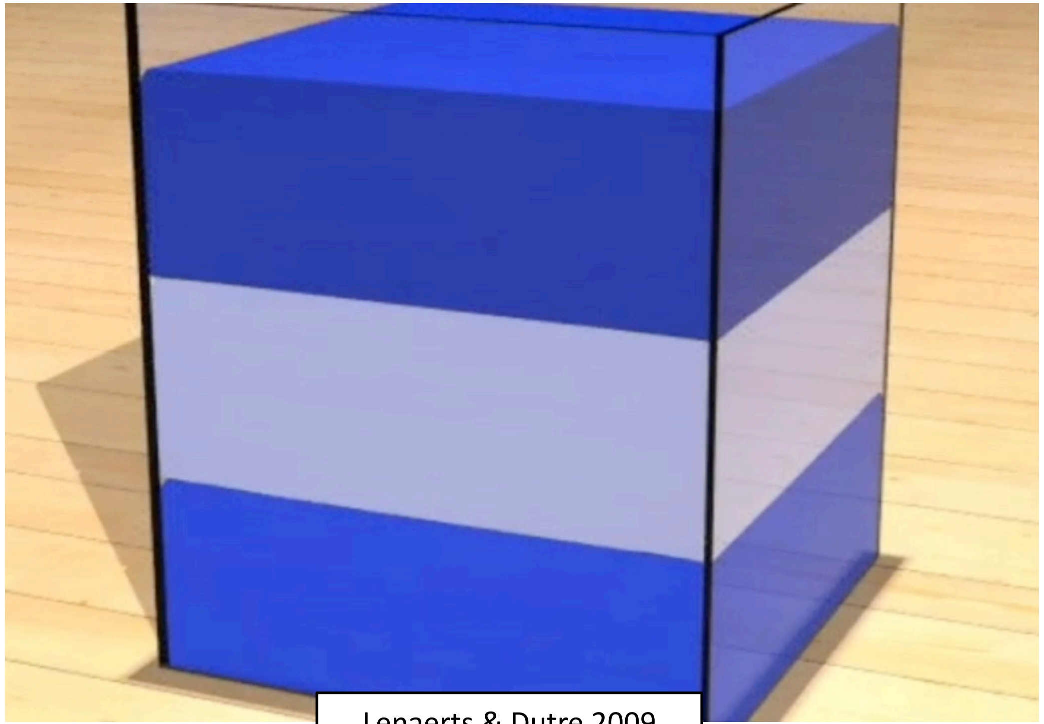
- Buoyancy emerges from individual rest densities



cgl SA

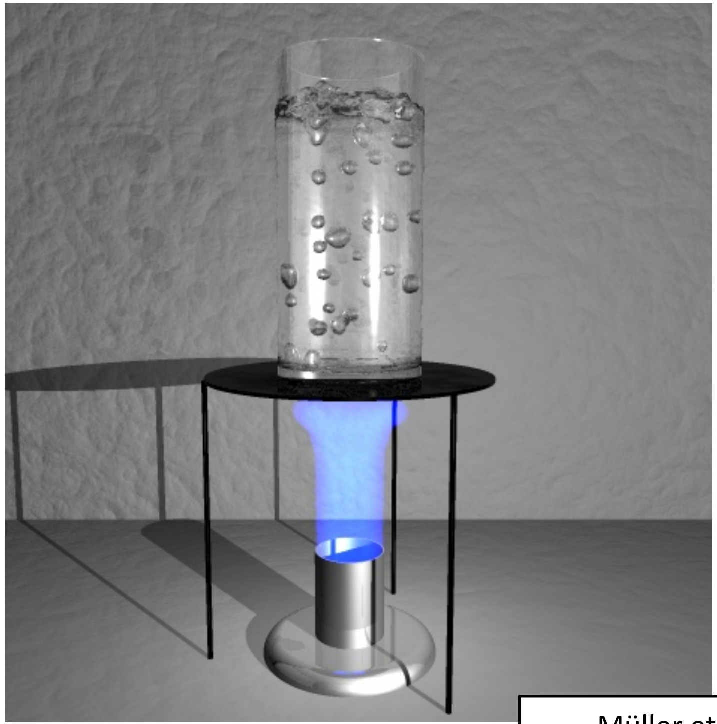
# My First Multi-fluid SPH Solver

Switching densities

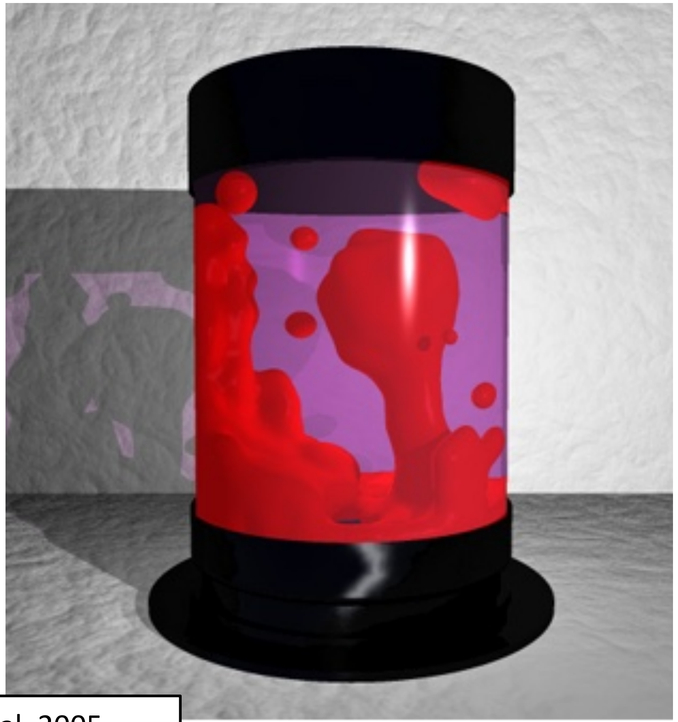


Lenaerts & Dutre 2009

Boiling

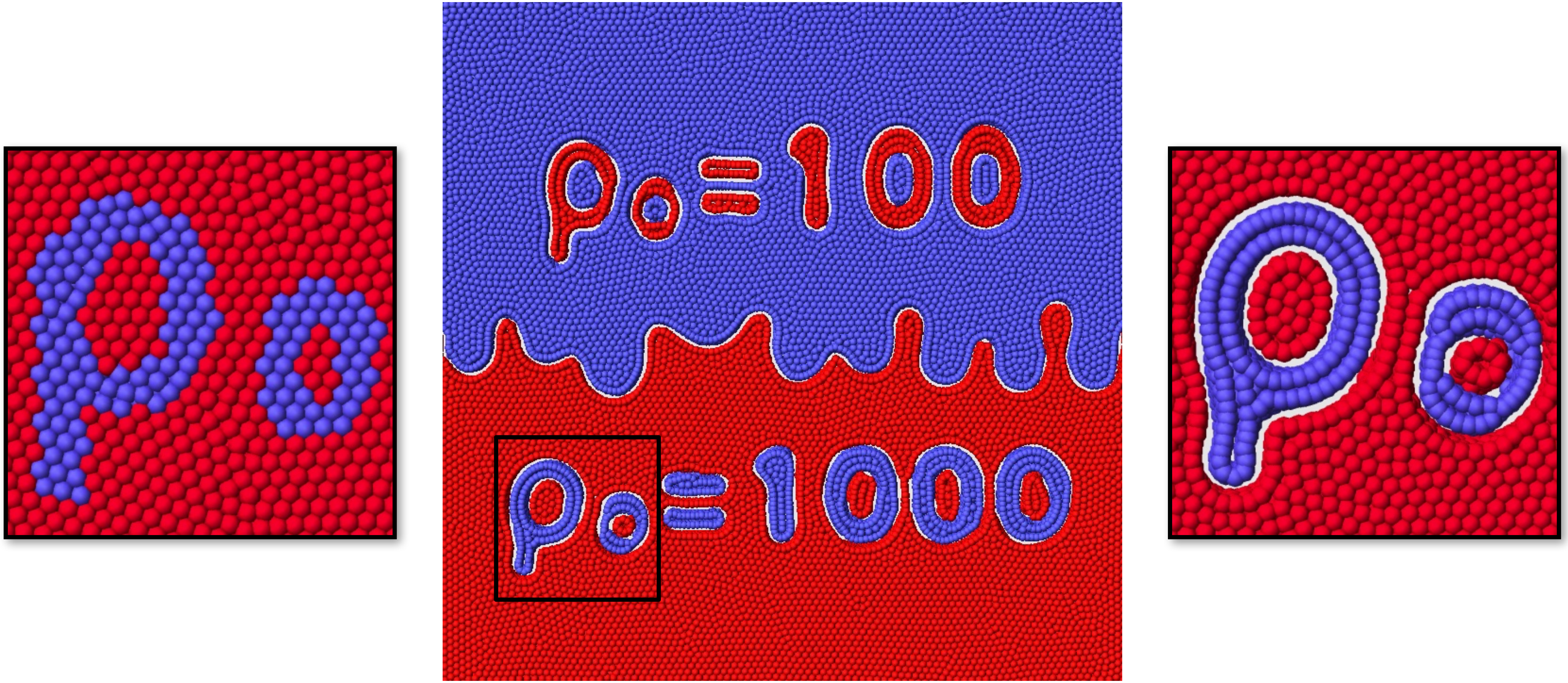


Lavalamp



Müller et al. 2005

# High Density Ratios

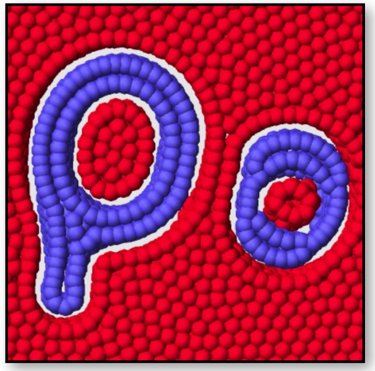


# High Density Ratios

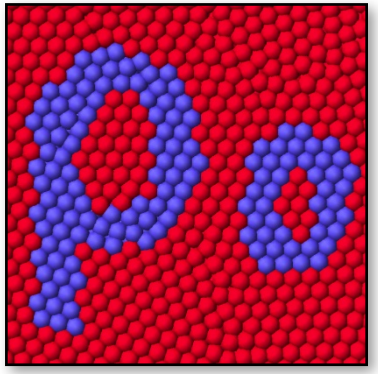
Solenthaler & Pajarola 2008

# Interface Discontinuities

- Standard SPH (SESPH)
  - Cannot handle discontinuities at interfaces
  - Results in spurious and unphysical interface tension
  - Large density differences lead to instability problems

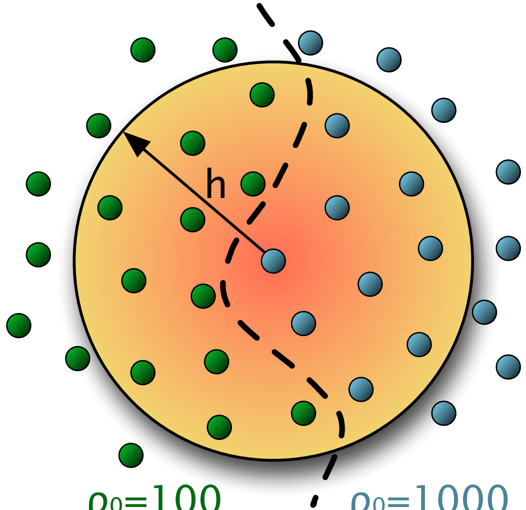


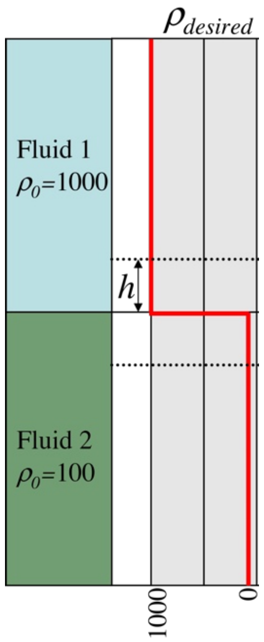
- Adapted SPH
  - Capture density discontinuities across interfaces
  - Stable simulations despite high density ratios
  - We need full control over behavior



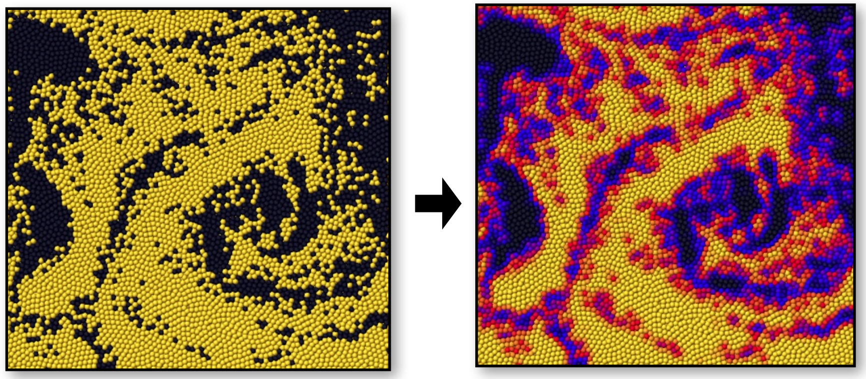
# Interface Discontinuities

- Problems near interfaces where rest densities and masses vary
- Falsified smoothed quantities



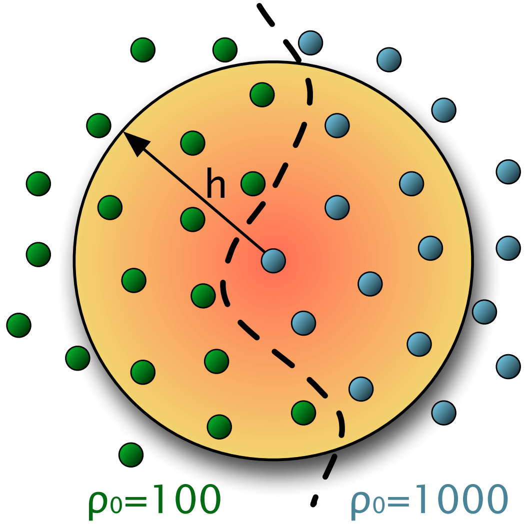

$$\rho_i = \sum_j m_j W_{ij}$$

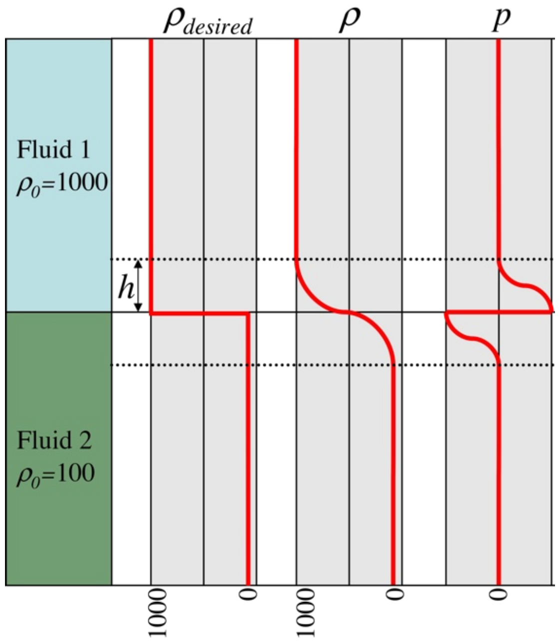


Color-coded density

# Interface Discontinuities

- Problems near interfaces where rest densities and masses vary
- Falsified smoothed quantities



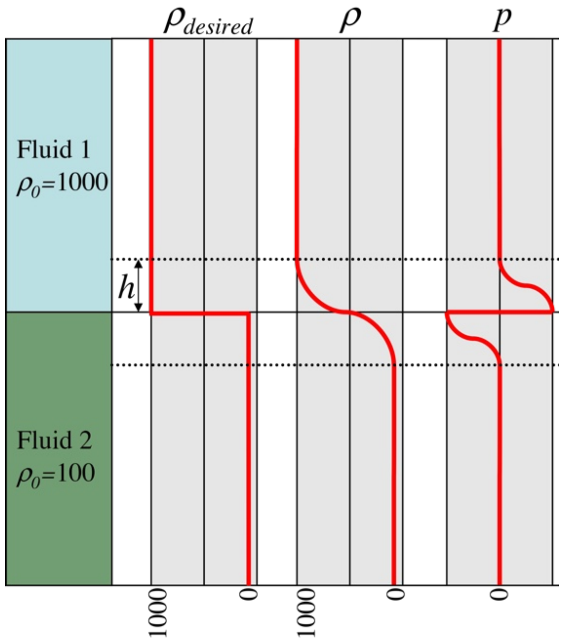

$$\frac{500}{1000} \Rightarrow p_i < 0$$

$$p_i = k_1 \left( \left( \frac{\rho_i}{\rho^0} \right)^{k_2} - 1 \right)$$

$$\frac{500}{100} \Rightarrow p_i > 0$$

# Adapted Density and Pressure

- Use number density $\delta_i = \sum_j W_{ij}$
- Adapted density of particle i given by $\tilde{\rho}_i = m_i \delta_i$
- Pressure computation using adapted density $\tilde{p}_i = k_1 \left( \left( \frac{\tilde{\rho}_i}{\rho^0} \right)^{k_2} - 1 \right)$



# Adapted Forces

- Derive adapted forces
- Substitute adapted density and pressure into the NS pressure term

$$\mathbf{F}^p = -\frac{\nabla \tilde{p}}{\delta}$$

- Apply SPH derivation to get adapted pressure force

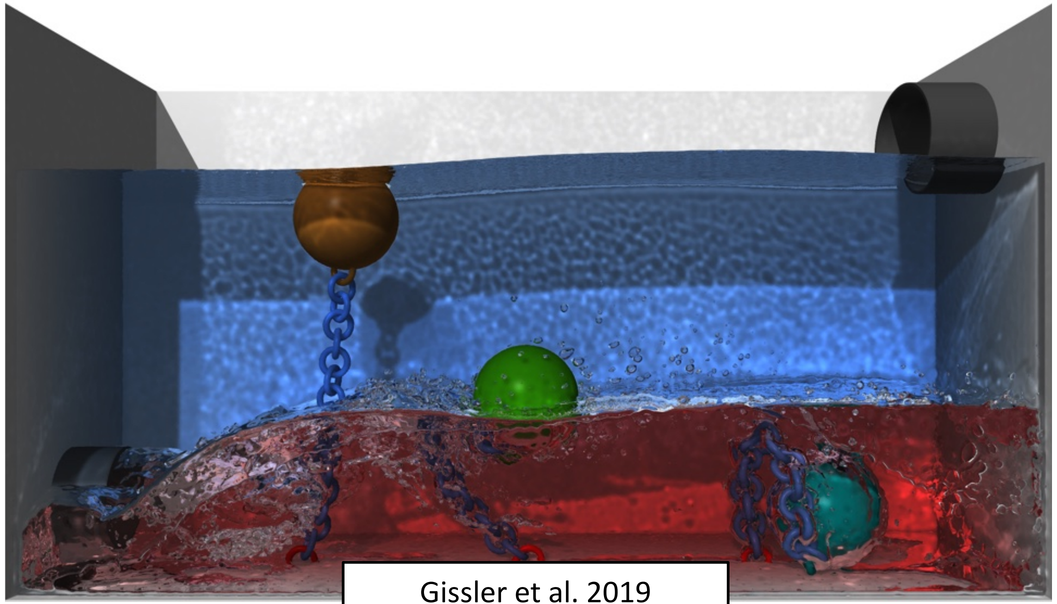
$$\mathbf{F}_i^p = -\sum_j \left( \frac{\tilde{p}_j}{{\delta_j}^2} + \frac{\tilde{p}_i}{{\delta_i}^2} \right) \nabla W_{ij}$$

- Similarly derivation of viscosity force

$$\mathbf{F}_i^v = \frac{1}{\delta_i} \sum_j \frac{\mu_i + \mu_j}{2} \frac{1}{\delta_j} (\mathbf{v}_j - \mathbf{v}_i) \nabla^2 W_{ij}$$

# Adapted SPH - Observations

- For a single phase fluid equations are identical to SESPH
- For multi-fluid simulations interface problems are eliminated
- No performance overhead
- Extended with incompressibility condition [Akinci et al. 12, Gissler et al. 19]



Gissler et al. 2019

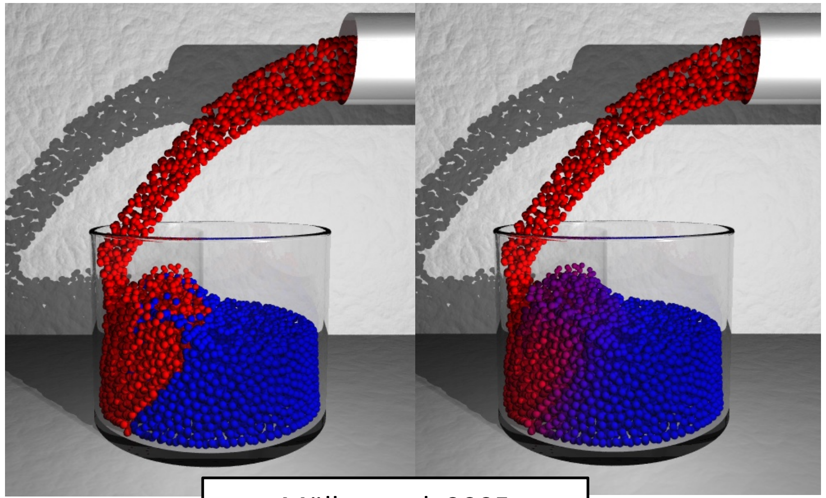
# Adapted SPH - Results

Solenthaler & Pajarola 2008

# Diffusion Effects

- Diffusion equation $\frac{\partial C}{\partial t} = \alpha \nabla^2 C$
- SPH equation $\frac{\partial C_i}{\partial t} = \alpha \sum_j m_j \frac{C_j - C_i}{\rho_j} \nabla^2 W_{ij}$

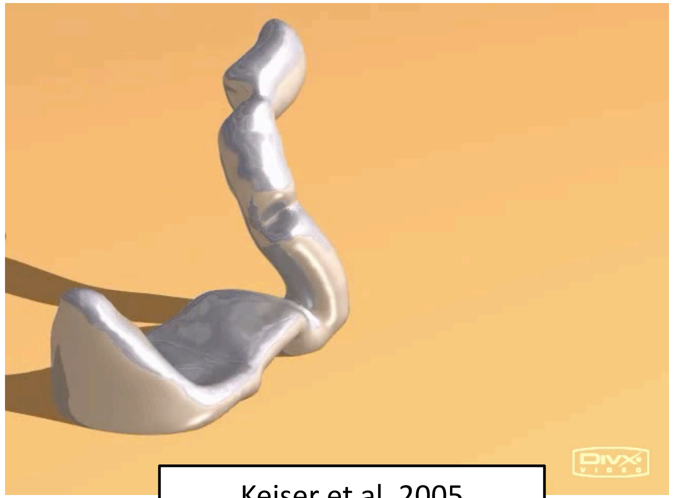
Color diffusion



Müller et al. 2005

Temperature diffusion (and phase changes)

Lenaerts & Dutre 2009

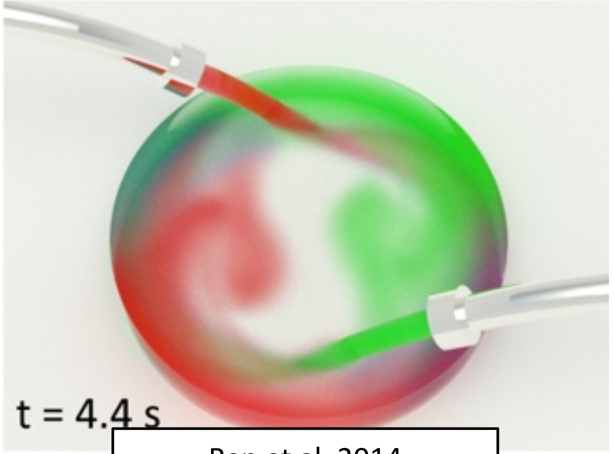


Keiser et al. 2005

# Complex Mixing Effects

- Previous work
  - Mixture is only caused by diffusion effects
  - Different phases move at the same bulk velocity as the mixture

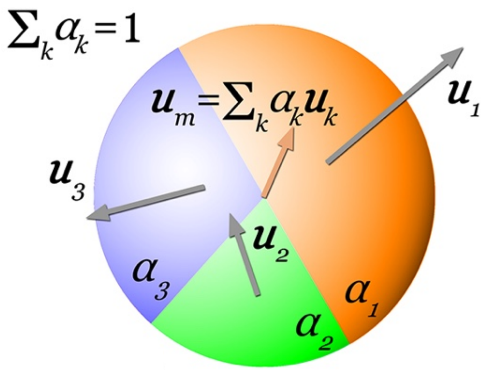
- SPH based mixture model [Ren et al. 2014]
  - Mixing and unmixing due to (relative) flow motion and force distribution
  - Dynamics of multi-fluid flow captured using mixture model
  - Spatial distribution of phases modeled using volume fraction (similar to [Müller et al. 05])
  - Drift velocities: Phase velocities relative to mixture average



Ren et al. 2014

# Mixture Model

- Phase:
  - Volume fraction $\alpha_k$ , $\sum_k \alpha_k = 1, \alpha_k \geq 0.$
  - Phase velocity v_k




- Mixture:
  - Mixture density (f($\alpha_k$))
  - Mixture velocity $\mathbf{v}_m$
- Continuity and momentum equations of the phases and mixture

➢ The nonuniform distribution of velocity fields will lead to changes in the volume fraction of each phase

➢ The drift velocities play a key role in this interaction mechanism

# Mixture Model

- Continuity equation of the mixture model $\frac{D\rho_m}{Dt} = \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m) = 0$

  $\rho_m$ mixture density $\rho_m = \sum_k \alpha_k \rho_k$

  $\alpha_k$ volume fraction of phase

  $\mathbf{v}_m$ mixture velocity (avg over all phases) $\mathbf{v}_m = \frac{1}{\rho_m \sum_k \alpha_k \rho_k \mathbf{v}_m}$

- Momentum equation for the mixture $\frac{D(\rho_m, \mathbf{v}_m)}{Dt} = -\nabla p + \nabla \cdot (\tau_m + \tau_{Dm}) + \rho_m \mathbf{g}$

  $\tau_m$ viscous stress tensor of the mixture

  $\tau_{Dm}$ diffusion tensor of the mixture (convective momentum transfer between phases)

- ➢ The nonuniform distribution of velocity fields will lead to changes in the volume fraction of each phase
- ➢ The drift velocities play a key role in this interaction mechanism

# Algorithm

*3 loops over all particles:*

1. Compute density and pressure with SPH

2. Compute drift velocity of each phase / particle

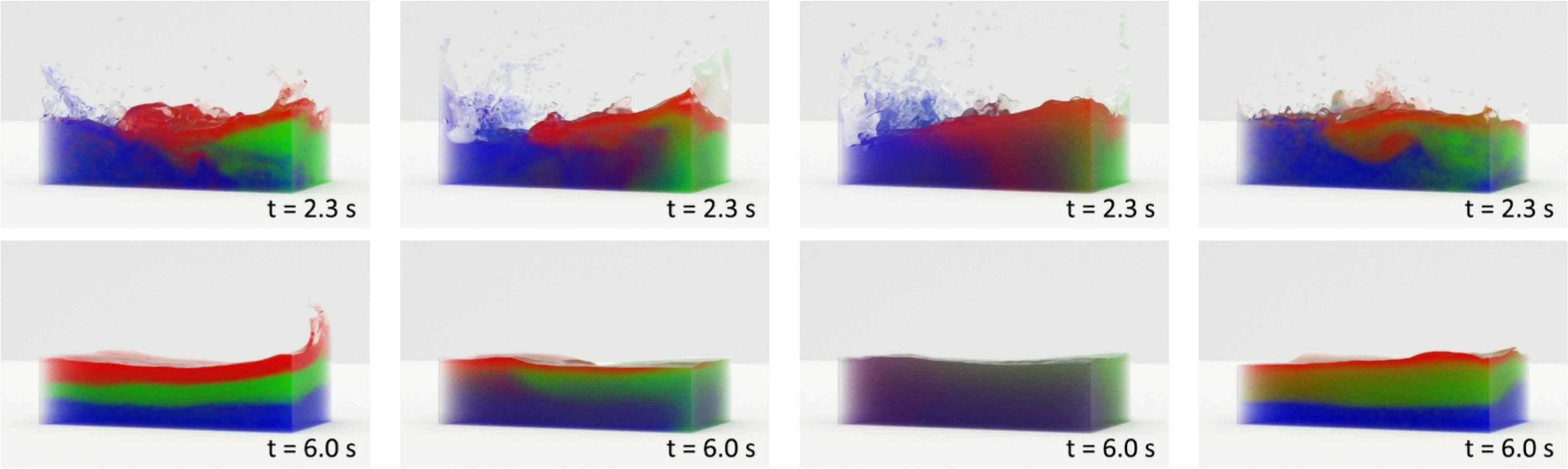
   Analytical expression of drift velocity, three terms defining
   - Slip velocity due to body forces
   - Pressure effects that cause fluid phases to move from high to low pressure regions
   - Brownian diffusion term representing phase drifting from high to low concentration

   Update diffusion tensor, advect volume fraction
   (using drift velocity)

3. Compute total force, advect particle

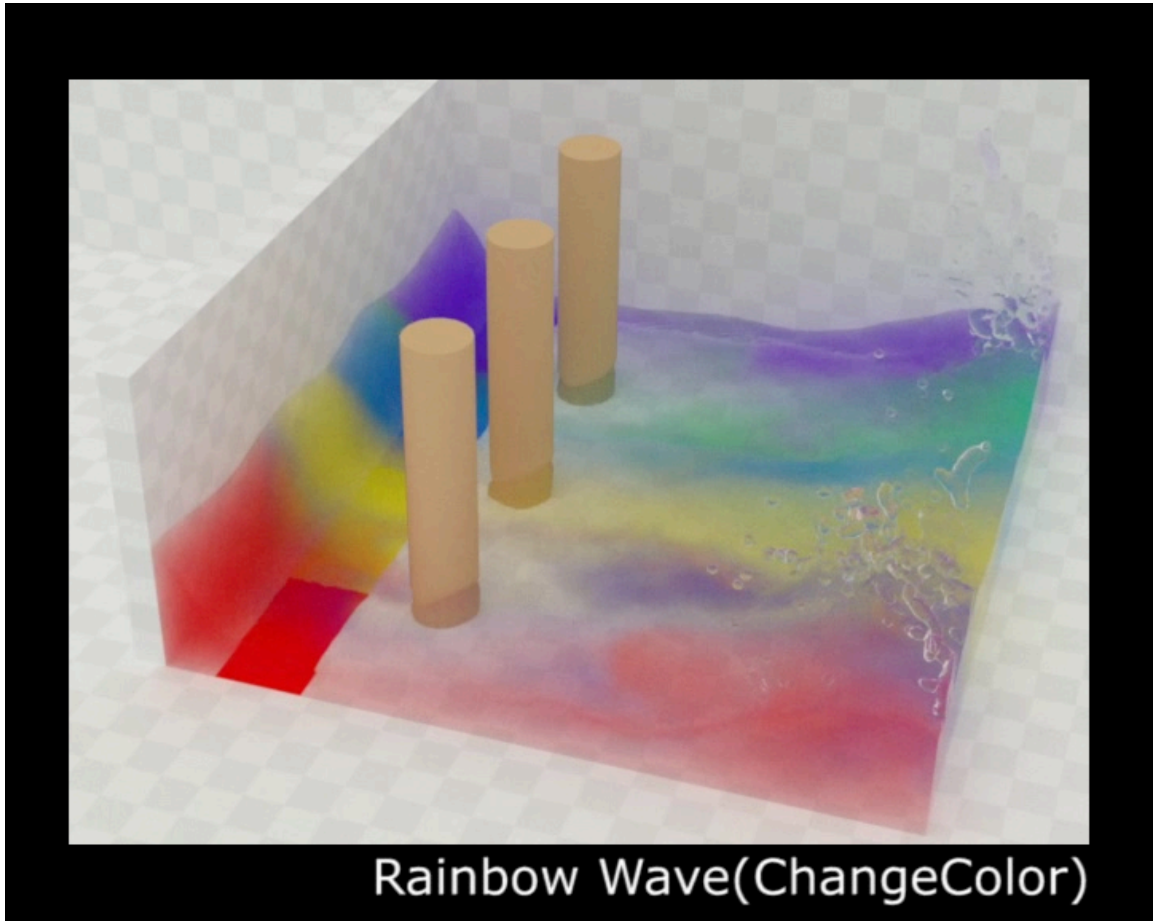
# Immiscible and Miscible Liquids

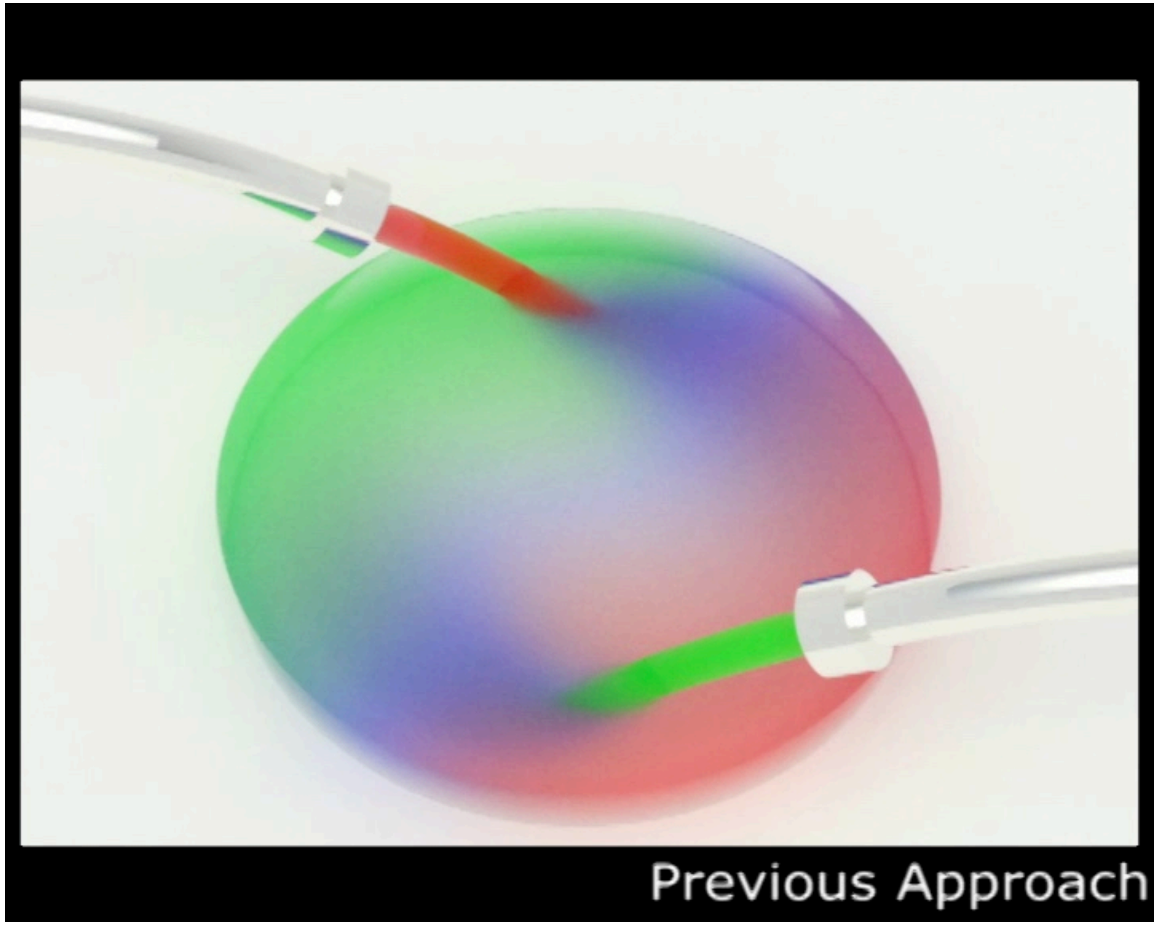
| Immiscible | Miscible, diffusion disabled | Miscible, diffusion enabled | Red / green miscible, immiscible with blue |
|---|---|---|---|




Ren et al. 2014

# More Results

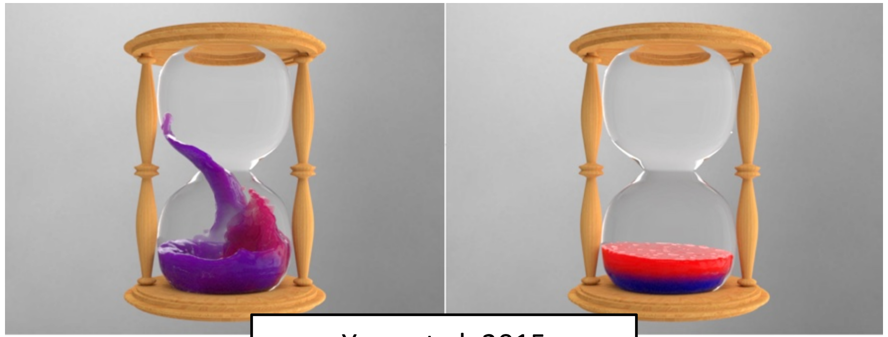


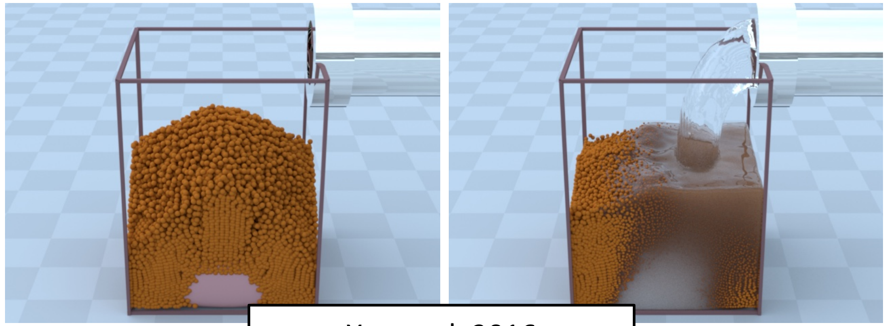

Ren et al. 2014

# Limitations and Extensions

- [Ren et al. 14] Uses WCSPH; a divergence-free velocity field cannot be directly integrated since neither the mixture nor phase velocities are zero, even if the material is incompressible
- [Yang et al. 15] Energy-based model using Cahn-Hilliard equation that describes phase separation -> incompressible flows
- [Yan et al. 16] Extension to fluid-solid interaction -> dissolution of solids, flows in porous media, interaction with elastics



Yang et al. 2015



Yan et al. 2016