Smoothed Particle Hydrodynamics

Techniques for the Physics Based Simulation of Fluids and Solids

Incompressibility

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SPH Fluid Solver

– Neighbor search
– Incompressibility
– Boundary handling

Outline

- Introduction
- Concepts
  - State equation
  - Iterative state equation
  - Pressure Poisson equation
- Current developments

Motivation

- Incompressibility is essential for a realistic fluid behavior
  - Less than 0.1% volume / density deviation in typical scenarios
- Inappropriate compression leads, e.g., to volume oscillations or volume loss
- Enforcing incompressibility significantly influences the performance
  - Simple approaches require small time steps
  - Expensive approaches work with large time steps

Approaches

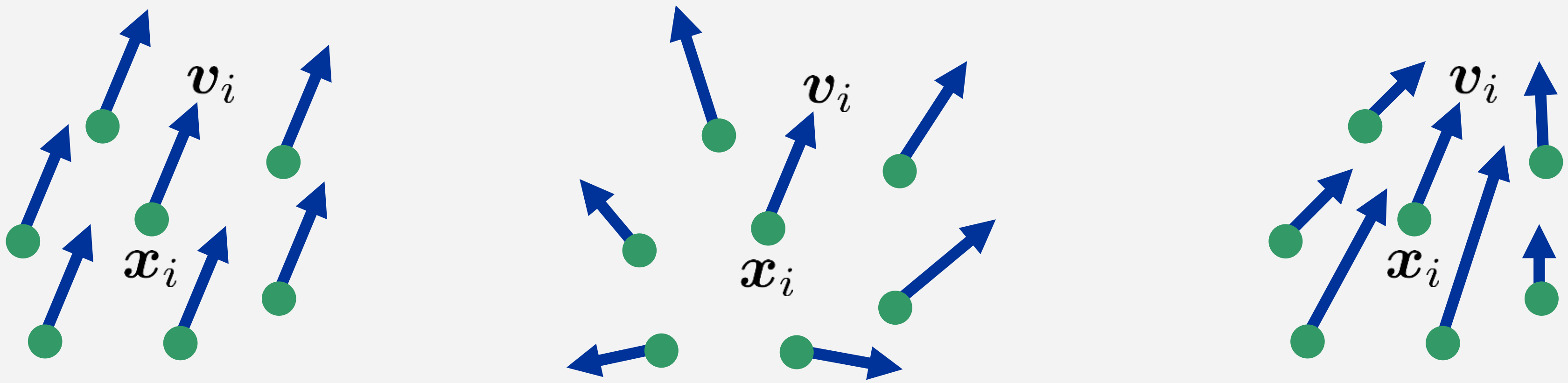
- Minimization of density / volume errors
  - Measure difference of actual and desired density
  - Compute pressure and pressure accelerations that reduce density / volume deviations
- Minimization of velocity divergence
  - Measure the divergence of the velocity field
  - Compute pressure and pressure accelerations that reduce the divergence of the velocity field

Typical Implementation

- Split pressure and non-pressure acceleration
  $\frac{\mathrm{D}\boldsymbol{v}(t)}{\mathrm{D}t} = -\frac{1}{\rho(t)}\nabla p(t) + \boldsymbol{a}^{\mathrm{nonp}}(t)$
- Predict velocity after non-pressure acceleration
  $\boldsymbol{v}^* = \boldsymbol{v}(t) + \Delta t \boldsymbol{a}^{\mathrm{nonp}}(t)$
- Compute pressure such that pressure acceleration either minimizes the divergence of $\boldsymbol{v}^*$ or the density error after advecting the samples with $\boldsymbol{v}^*$
- Update velocity $\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}^* - \Delta t \frac{1}{\rho(t)}\nabla p(t)$
  - Minimized density error / divergence at advected samples

Density Invariance vs. Velocity Divergence

- Continuity equation: $\frac{\mathrm{D}\rho_i}{\mathrm{D}t} = -\rho_i \nabla \cdot \boldsymbol{v}_i$
  - Time rate of change of the density is related to the divergence of the velocity



$\frac{\mathrm{D}\rho_i}{\mathrm{D}t} = -\rho_i \nabla \cdot \boldsymbol{v}_i = 0$ $\qquad$ $\frac{\mathrm{D}\rho_i}{\mathrm{D}t} = -\rho_i \nabla \cdot \boldsymbol{v}_i < 0$ $\qquad$ $\frac{\mathrm{D}\rho_i}{\mathrm{D}t} = -\rho_i \nabla \cdot \boldsymbol{v}_i > 0$

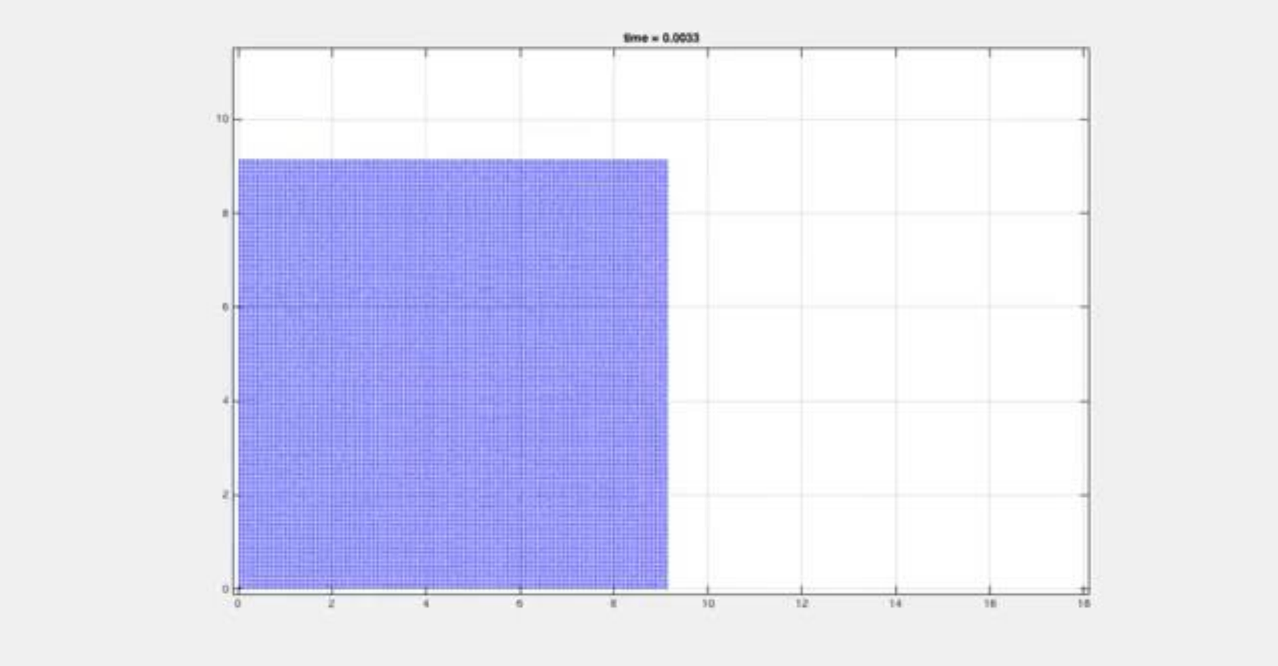
$\nabla \cdot \boldsymbol{v}_i = 0$ $\qquad$ $\nabla \cdot \boldsymbol{v}_i > 0$ $\qquad$ $\nabla \cdot \boldsymbol{v}_i < 0$

Density Invariance vs. Velocity Divergence

- Density invariance
  - Measure and minimize density deviations
- Velocity divergence
  - Measure and minimize the divergence of the velocity field
  - Zero velocity divergence corresponds to zero density change over time $-\rho_i \nabla \cdot \boldsymbol{v}_i = \frac{\mathrm{D}\rho_i}{\mathrm{D}t} = 0$, i.e. the initial density does not change over time
  - Notion of density is not required

Challenges

- Minimizing density deviations can result in volume oscillations
  - Density error is going up and down
  - Erroneous fluid dynamics
  - Only very small density deviations are tolerable, e.g. 0.1%



https://www.youtube.com/watch?v=hAPO0xBp5WU

Challenges

- Minimizing the velocity divergence can result in volume loss
  - Divergence errors result in density drift
  - No notion of actual density

[Zhu, Lee, Quigley, Fedkiw, ACM SIGGRAPH 2015]

SPH Graphics Research - Incompressibility

- State equation
  - [Becker 2007]
- Iterative state equation
  - PCISPH [Solenthaler 2009],
    LPSPH [He 2012],
    PBF [Macklin 2013]
- Pressure Poisson equation
  - IISPH [Ihmsen 2013],
    DFSPH [Bender 2015],
    [Cornelis 2018]

Incompressibility – Applications

- Fluids
- Elastic solids
- Rigid bodies
- Monolithic solvers with unified representations


Valley
up to 38M fluid particles interacting with more than
650 rigid bricks, highly viscous mud and an elastic tree
[Gissler et al., presented at ACM SIGGRAPH 2019]

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State Equation SPH (SESPH)

- Compute pressure from the density deviation
  locally with one equation for each sample / particle
- Compute pressure acceleration

State Equations

– Pressure is proportional to density error

  – E.g. $p_i = k(\frac{\rho_i}{\rho_0} - 1)$ or $p_i = k(\rho_i - \rho_0)$

    – Referred to as compressible SPH

  – $p_i = k\left(\left(\frac{\rho_i}{\rho_0}\right)^7 - 1\right)$

    – Referred to as weakly compressible SPH

Pressure values in SPH implementations should always be non-negative.

SESPH – State Equation SPH Fluid Solver

for all *particle* i **do**
  find neighbors j
for all *particle* i **do**
  $\rho_i = \sum_j m_j W_{ij}$
  $p_i = k(\frac{\rho_i}{\rho_0} - 1)$

Compute pressure with a state equation

for all *particle* i **do**
  $\boldsymbol{a}_i^{\text{nonp}} = \nu \nabla^2 \boldsymbol{v}_i + \boldsymbol{g}$
  $\boldsymbol{a}_i^{\text{p}} = -\frac{1}{\rho_i} \nabla p_i$
  $\boldsymbol{a}_i(t) = \boldsymbol{a}_i^{\text{nonp}} + \boldsymbol{a}_i^{\text{p}}$
for all *particle* i **do**
  $\boldsymbol{v}_i(t + \Delta t) = \boldsymbol{v}_i(t) + \Delta t \boldsymbol{a}_i(t)$
  $\boldsymbol{x}_i(t + \Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t + \Delta t)$

SESPH - Discussion

- Compression results in pressure
- Pressure gradients result in accelerations from high to low density
- Simple computation, small time steps
- Larger stiffness $\rightarrow$ less compressibility $\rightarrow$ smaller time step
- Stiffness constant k does not govern the pressure, but the compressibility of the fluid

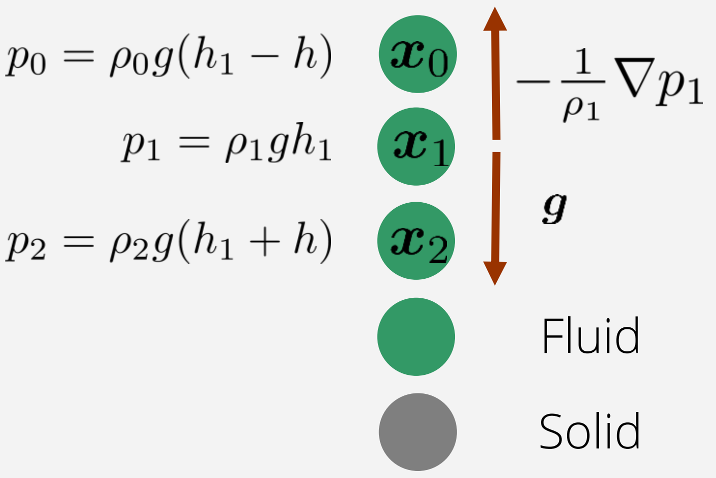
Stiffness Constant – 1D Illustration

- Gravity cancels pressure acceleration

$$\boldsymbol{g} = -\boldsymbol{a}_i^{\mathrm{p}} = \frac{1}{\rho_i}\nabla p_i = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla W_{ij}$$

$$= \sum_j m_j \left(\frac{k(\rho_i - \rho_0)}{\rho_i^2} + \frac{k(\rho_j - \rho_0)}{\rho_j^2}\right)\nabla W_{ij}$$

- Differences between p_i and p_j are independent from k
- Smaller k results in larger density error $\rho_i - \rho_0$ to get the required pressure



A 1D fluid under gravity at rest

SESPH with Splitting

- Split pressure and non-pressure accelerations
  - Non-pressure acceleration $\boldsymbol{a}_i^{\text{nonp}}$
  - Predicted velocity $\boldsymbol{v}_i^* = \boldsymbol{v}_i(t) + \Delta t \boldsymbol{a}_i^{\text{nonp}}$
  - Predicted position $\boldsymbol{x}_i^* = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i^*$
  - Predicted density $\rho_i^*(\boldsymbol{x}_i^*)$
  - Pressure p from predicted density ρ_i^*
  - Pressure acceleration $\boldsymbol{a}_i^{\text{p}}$
  - Final velocity and position $\boldsymbol{v}_i(t+\Delta t) = \boldsymbol{v}_i^* + \Delta t \boldsymbol{a}_i^{\text{p}} = \boldsymbol{v}_i^* - \Delta t \frac{1}{\rho_i^*} \nabla p_i$

    $\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t+\Delta t)$

SESPH with Splitting

for all *particle i* **do**

find neighbors j

for all *particle i* **do**

$\boldsymbol{a}_i^{\mathrm{nonp}} = \nu \nabla^2 \boldsymbol{v}_i + \boldsymbol{g}$

$\boldsymbol{v}_i^* = \boldsymbol{v}_i(t) + \Delta t \boldsymbol{a}_i^{\mathrm{nonp}}$

for all *particle i* **do**

$\rho_i^* = \sum_j m_j W_{ij} + \Delta t \sum_j m_j (\boldsymbol{v}_i^* - \boldsymbol{v}_j^*) \nabla W_{ij}$ Density at predicted positions

$p_i = k(\frac{\rho_i^*}{\rho_0} - 1)$ Pressure at predicted positions

for all *particle i* **do**

$\boldsymbol{a}_i^{\mathrm{p}} = -\frac{1}{\rho_i} \nabla p_i$

for all *particle i* **do**

$\boldsymbol{v}_i(t+\Delta t) = \boldsymbol{v}_i^* + \Delta t \boldsymbol{a}_i^{\mathrm{p}}$

$\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t+\Delta t)$

Differential Density Update

- Density at advected positions is often approximated without advecting the samples
- Continuity equation and time discretization

$$\frac{\mathrm{D}\rho_i}{\mathrm{D}t} = -\rho_i \nabla \cdot \boldsymbol{v}_i \qquad \frac{\rho_i^* - \rho_i(t)}{\Delta t} = -\rho_i \nabla \cdot \boldsymbol{v}_i^*$$

- SPH discretization

$$\frac{\rho_i^* - \sum_j m_i W_{ij}}{\Delta t} = -\rho_i \left(-\frac{1}{\rho_i} \sum_j m_j (\boldsymbol{v}_i^* - \boldsymbol{v}_j^*) \nabla W_{ij} \right)$$

- Predicted density due to the divergence of $\boldsymbol{v}_i^*$

$$\rho_i^* = \sum_j m_i W_{ij} + \Delta t \sum_j m_j (\boldsymbol{v}_i^* - \boldsymbol{v}_j^*) \nabla W_{ij}$$

Approximate density at predicted positions: $\boldsymbol{x}_i^* = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i^*$

SESPH with Splitting - Discussion

- Consider competing accelerations
- Take effects of non-pressure accelerations into account when computing the pressure acceleration
- Incompressibility has highest priority

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Iterative SESPH with Splitting

- Pressure accelerations are iteratively refined
  - Non-pressure acceleration $\boldsymbol{a}_i^{\text{nonp}}$
  - Predicted velocity $\boldsymbol{v}_i^* = \boldsymbol{v}_i(t) + \Delta t \boldsymbol{a}_i^{\text{nonp}}$
  - Iterate until convergence
    - Density from predicted position $\rho_i^*(\boldsymbol{x}_i, \boldsymbol{v}_i^*)$
    - Pressure from predicted density p_i
    - Pressure acceleration $\boldsymbol{a}_i^{\text{p}}$
    - Refine predicted velocity $\boldsymbol{v}_i^* = \boldsymbol{v}_i^* + \Delta t \boldsymbol{a}_i^{\text{p}}$
  - Final velocity and position $\boldsymbol{v}_i(t + \Delta t) = \boldsymbol{v}_i^*$

$$\boldsymbol{x}_i(t + \Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t + \Delta t)$$

Iterative SESPH with Splitting - Motivation

- Iterative update is parameterized by a desired density error
- Provides a fluid state with a guaranteed density error
- Stiffness parameter and form of the state equation govern the convergence rate

Iterative SESPH with Splitting

for all *particle* i **do**
  find neighbors j

for all *particle* i **do**
  $\boldsymbol{a}_i^{\text{nonp}} = \nu \nabla^2 \boldsymbol{v}_i + \boldsymbol{g} \quad ; \quad \boldsymbol{v}_i^* = \boldsymbol{v}_i(t) + \Delta t \boldsymbol{a}_i^{\text{nonp}}$

repeat
  for all *particle* i **do**
    $\rho_i^* = \sum_j m_j W_{ij} + \Delta t \sum_j m_j (\boldsymbol{v}_i^* - \boldsymbol{v}_j^*) \nabla W_{ij}$
    $p_i = k(\frac{\rho_i^*}{\rho_0} - 1)$

  for all *particle* i **do**
    $\boldsymbol{v}_i^* = \boldsymbol{v}_i^* - \Delta t \frac{1}{\rho_i^*} \nabla p_i$

until $\rho_i^* - \rho_0 < \eta$ user-defined density error

for all *particle* i **do**
  $\boldsymbol{v}_i(t + \Delta t) = \boldsymbol{v}_i^* \quad ; \quad \boldsymbol{x}_i(t + \Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t + \Delta t)$

Iterative SESPH - Variants

- Different quantities are accumulated
  - Velocity changes (local Poisson SPH LPSPH)
  - Pressure (predictive-corrective SPH PCISPH) [Solenthaler 2009]
    - Advantageous, if pressure is required for other computations
  - Distances (position-based fluids PBF)
    - $\Delta \boldsymbol{x}_i = -\frac{1}{\rho_0} \sum_j (\frac{p_i}{\beta_i} + \frac{p_j}{\beta_j}) \nabla W_{ij}$
- Different EOS and stiffness constants are used
  - $\tilde{p}_i = k(\rho_i - \rho_0)$ with $k = \frac{\rho_i^* r_i^2}{2\rho_0 \Delta t^2}$ in local Poisson SPH
  - $p_i = k(\rho_i - \rho_0)$ with $k = \frac{\rho_0^2}{2m_i^2 \cdot \Delta t^2 (\sum_j \nabla W_{ij}^0 \cdot \sum_j \nabla W_{ij}^0 + \sum_j (\nabla W_{ij}^0 \cdot \nabla W_{ij}^0))}$ in PCISPH
  - $\tilde{p}_i = k(\frac{\rho_i}{\rho_0} - 1)$ with $k = 1$ in PBF

Predictive-Corrective Incompressible SPH - PCISPH

- Goal: Computation of pressure accelerations $\boldsymbol{a}_i^{\mathrm{p}}$ that result in rest density ρ_0 at all particles
- Formulation: Density at the next time step should equal the rest density

Desired density | Current density | Density change due to predicted velocity | Density change due to unknown pressure acceleration

$$\rho(t+\Delta t) = \rho_0 = \underbrace{\sum_j m_i W_{ij} + \Delta t \sum_j m_j (\boldsymbol{v}_i^* - \boldsymbol{v}_j^*) \nabla W_{ij}}_{\rho_i^*} + \Delta t \sum_j m_j (\Delta t \boldsymbol{a}_i^{\mathrm{p}} - \Delta t \boldsymbol{a}_j^{\mathrm{p}}) \nabla W_{ij}$$

Discretized continuity equation

PCISPH - Assumptions

– Simplifications to get one equation with one unknown:
  – Equal pressure at all neighboring samples

$$\boldsymbol{a}_i^{\mathrm{p}} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \nabla W_{ij} \approx -m_i \frac{2p_i}{\rho_0^2} \sum_j \nabla W_{ij}$$

$$\rho_0 - \rho_i^* = \Delta t^2 \sum_j m_j \left(-m_i \frac{2p_i}{\rho_0^2} \sum_j \nabla W_{ij} + m_j \frac{2p_j}{\rho_0^2} \sum_k \nabla W_{jk} \right) \nabla W_{ij}$$

Unknown pressures p_i and p_j

  – For sample *j*, only consider the contribution from *i*

$$\rho_0 - \rho_i^* = \Delta t^2 \sum_j m_j \left(-m_i \frac{2p_i}{\rho_0^2} \sum_j \nabla W_{ij} + m_i \frac{2p_i}{\rho_0^2} \nabla W_{ji} \right) \nabla W_{ij}$$

Unknown pressure p_i

$$\rho_0 - \rho_i^* = \Delta t^2 m_i^2 \frac{2p_i}{\rho_0^2} \sum_j \left(-\sum_j \nabla W_{ij} - \nabla W_{ij} \right) \nabla W_{ij} = -\Delta t^2 m_i^2 \frac{2p_i}{\rho_0^2} \left(\sum_j \nabla W_{ij} \cdot \sum_j \nabla W_{ij} + \sum_j (\nabla W_{ij} \cdot \nabla W_{ij}) \right)$$

PCISPH - Solution

– Solve for unknown pressure:

$$\rho_0 - \rho_i^* = -\Delta t^2 m_i^2 \frac{2p_i}{\rho_0^2} \left(\sum_j \nabla W_{ij} \cdot \sum_j \nabla W_{ij} + \sum_j (\nabla W_{ij} \cdot \nabla W_{ij}) \right)$$

$$p_i = \frac{\rho_0^2}{2\Delta t^2 m_i^2 (\sum_j \nabla W_{ij} \cdot \sum_j \nabla W_{ij} + \sum_j (\nabla W_{ij} \cdot \nabla W_{ij}))} (\rho_i^* - \rho_0) \qquad (\ p_i = k(\rho_i^* - \rho_0)\ )$$

Intuition: This pressure causes pressure accelerations that cause velocity changes that correspond to a divergence that results in rest density at the sample.

$$\rho(t + \Delta t) = \rho_0 = \rho_i^* + \Delta t \sum_j m_j (\Delta t \boldsymbol{a}_i^{\mathrm{p}} - \Delta t \boldsymbol{a}_j^{\mathrm{p}}) \nabla W_{ij}$$

PCISPH - Discussion

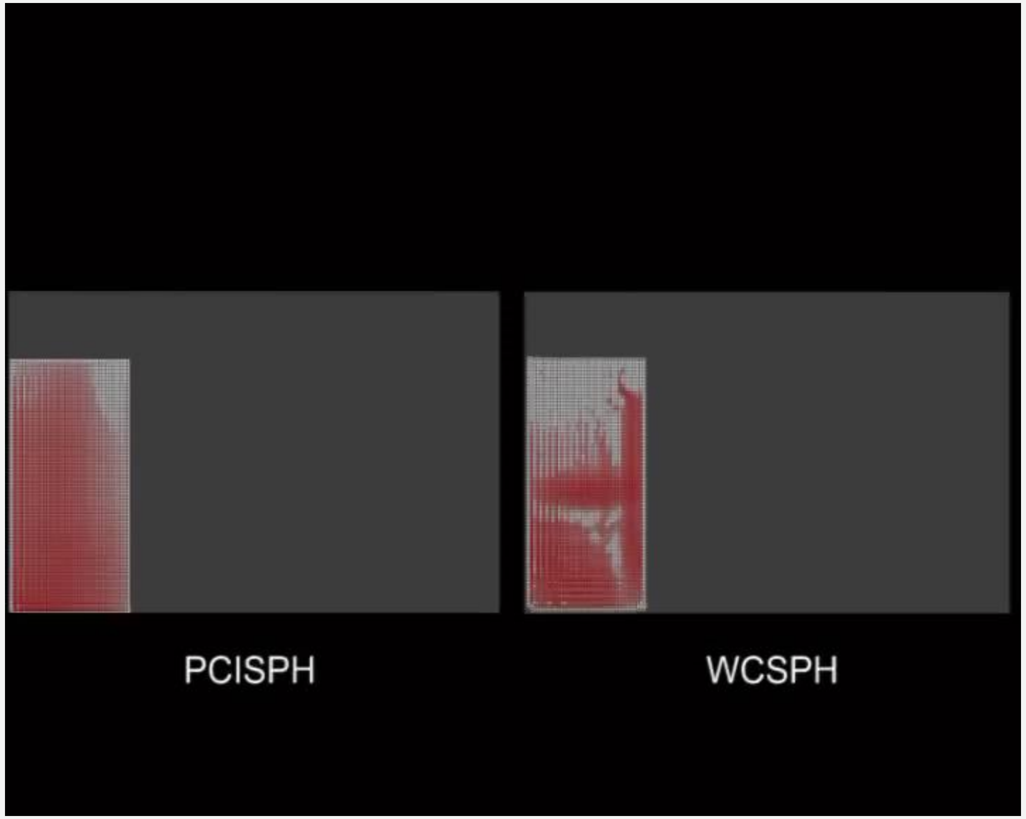
- Pressure is computed with a state equation $p_i = k_i(\rho_i^* - \rho_0)$
- k_i is not user-defined
- Instead, an optimized value k_i is derived and used
- Pressure is iteratively refined

PCISPH - Performance

- Typically three to five iterations for density errors between 0.1% and 1%
- Speed-up factor over non-iterative SESPH up to 50
  - More computations per time step compared to SESPH
  - Significantly larger time step than in SESPH
  - Speed-up dependent on scenario
- Non-linear relation between time step and iterations
  - Largest possible time step does not necessarily lead to an optimal overall performance

Comparison

- PCISPH [Solenthaler 2009]
  - Iterative pressure computation
  - Large time step
- WCSPH [Becker and Teschner 2007]
  - Efficient to compute
  - Small time step
- Computation time for the PCISPH scenario is 20 times shorter than WCSPH



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Introduction

- Pressure causes pressure accelerations that cause velocity change that cause displacements such that particles have rest density
- Projection schemes solve a linear system to compute the respective pressure field
  - PCISPH uses simplifications to compute pressure per particle from one equation. Solving a linear system is avoided.

Derivation

$$\frac{\mathrm{D}\boldsymbol{v}(t)}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p(t) + \boldsymbol{a}^{\mathrm{nonp}}(t)$$

Velocity change per time step due to pressure acceleration and non-pressure acceleration

$$\boldsymbol{v}^* = \boldsymbol{v}(t) + \Delta t \boldsymbol{a}^{\mathrm{nonp}}(t)$$

Predicted velocity after non-pressure acceleration

$$\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}^* - \Delta t \frac{1}{\rho}\nabla p(t)$$

Velocity after all accelerations

$$\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}^* = -\Delta t \frac{1}{\rho}\nabla p(t)$$

Velocity change due to pressure acceleration

$$\nabla \cdot (\boldsymbol{v}^* - \boldsymbol{v}(t+\Delta t)) = \nabla \cdot \left(\Delta t \frac{1}{\rho}\nabla p(t)\right)$$

Divergence of the velocity change due to pressure acceleration

Derivation

$$\nabla \cdot (\boldsymbol{v}^* - \boldsymbol{v}(t+\Delta t)) = \nabla \cdot \left(\Delta t \frac{1}{\rho} \nabla p(t)\right)$$

$$\nabla \cdot \boldsymbol{v}^* - \nabla \cdot \boldsymbol{v}(t+\Delta t) = \nabla \cdot \left(\Delta t \frac{1}{\rho} \nabla p(t)\right)$$

Constraint: $\nabla \cdot \boldsymbol{v}(t+\Delta t) = 0$

Divergence of the final velocity field should be zero, i.e. no density change per time

$$\nabla \cdot \boldsymbol{v}^* = -\nabla \cdot (\Delta t \boldsymbol{a}^{\mathrm{p}})$$

Divergence of the velocity change due to pressure acceleration should cancel the divergence of the predicted velocity

$$\rho \nabla \cdot \boldsymbol{v}^* = \Delta t \nabla^2 p(t)$$

Pressure Poisson equation with unknown pressure

Density Invariance vs. Velocity Divergence

- Pressure Poisson equation PPE that minimizes the velocity divergence: $\Delta t \nabla^2 p(t) = \rho \nabla \cdot \boldsymbol{v}^*$
- PPE that minimizes the density error: $\Delta t \nabla^2 p(t) = \frac{\rho_0 - \rho^*}{\Delta t}$
- Derivation:

$$\frac{\mathrm{D}\rho(t+\Delta t)}{\mathrm{D}t} + \rho(t+\Delta t)\nabla \cdot \boldsymbol{v}(t+\Delta t) = 0$$

Discretized continuity equation at time $t + \Delta t$

Constraint: $\rho(t+\Delta t) = \rho_0$

$$\frac{\rho_0 - \rho(t)}{\Delta t} + \rho_0 \nabla \cdot \left(\boldsymbol{v}^* - \Delta t \frac{1}{\rho_0} \nabla p(t) \right) = 0$$

$$\frac{\rho_0 - (\rho(t) - \Delta t \rho_0 \nabla \cdot \boldsymbol{v}^*)}{\Delta t} - \Delta t \nabla^2 p(t) = 0$$

$$\rho^* = \rho(t) - \Delta t \rho_0 \nabla \cdot \boldsymbol{v}^*$$

Predicted density after sample advection with $\boldsymbol{v}^*$

Interpretation of PPE Forms

- Velocity divergence: $-\Delta t \frac{1}{\rho} \nabla^2 p = -\nabla \cdot \boldsymbol{v}^*$
  - Pressure p causes a pressure acceleration $-\frac{1}{\rho}\nabla p$ that causes a velocity change $-\Delta t \frac{1}{\rho}\nabla p$ whose divergence $\nabla \cdot (-\Delta t \frac{1}{\rho}\nabla p)$ cancels the divergence $\nabla \cdot \boldsymbol{v}^*$ of the predicted velocity, i.e. $\nabla \cdot \boldsymbol{v}^* + \nabla \cdot (-\Delta t \frac{1}{\rho}\nabla p) = 0$
- Density invariance: $-\Delta t \nabla^2 p = -\frac{\rho_0 - \rho^*}{\Delta t}$
  - The divergence $\nabla \cdot (-\Delta t \frac{1}{\rho}\nabla p)$ multiplied with density ρ is a density change per time that cancels the predicted density error per time $\frac{\rho_0 - \rho^*}{\Delta t}$, i.e. $\frac{\rho_0 - \rho^*}{\Delta t} + \rho \nabla \cdot (-\Delta t \frac{1}{\rho} \nabla^2 p) = 0$

PPE Solver

- Linear system with unknown pressure values $\boldsymbol{Ap} = \boldsymbol{s}$
  - One equation per particle $(\boldsymbol{Ap})_i = s_i \quad (\Delta t < \nabla^2 p_i >= \frac{\rho_0 - <\rho_i^*>}{\Delta t})$

<*A*> is a discretized form of *A*

- Iterative solvers
  - Conjugate Gradient
  - Relaxed Jacobi
- Fast computation per iteration
  - Few non-zero entries in each equation
  - Matrix-free implementations
  - Very few information per particle

PPE Solver

- Very large time steps
- Convergence dependent on the formulation
  - SPH discretization of $\nabla^2 p$
  - Source term (velocity divergence or density invariance)
- Accuracy issues
  - Volume drift for velocity divergence
  - Oscillations for density invariance

PPE Discretization

- Implicit incompressible SPH (IISPH) [Ihmsen et al. 2014]
  - PPE with density invariance as source term: $\Delta t^2 \nabla^2 p = \rho_0 - \rho^*$
  - Computation of ρ_i^*:
    $\rho_i^* = \rho_i + \Delta t \sum_j m_j \boldsymbol{v}_{ij}^* \nabla W_{ij}$ with $\boldsymbol{v}_i^* = \boldsymbol{v}_i + \Delta t \boldsymbol{a}_i^{\text{nonp}}$
  - Computation of $\Delta t^2 \nabla^2 p_i$:
    $\Delta t^2 \nabla^2 p_i = -\Delta t \rho_i \nabla \cdot (\Delta t \boldsymbol{a}_i^{\text{p}}) = \Delta t^2 \sum_j m_j \left(\boldsymbol{a}_i^{\text{p}} - \boldsymbol{a}_j^{\text{p}} \right) \cdot \nabla W_{ij}$
    with
    $\boldsymbol{a}_i^{\text{p}} = -\frac{1}{\rho_i} \nabla p_i = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij}$

PPE System - IISPH

– PPE $$\Delta t^2 \nabla^2 p_i = \rho_0 - \rho_i^*$$

density change due to pressure accelerations

negative of the predicted density error

– Discretized PPE

– System: $\boldsymbol{A}\boldsymbol{p} = \boldsymbol{s}$

– Per particle: $$\underbrace{\Delta t^2 \sum_j m_j \left(\boldsymbol{a}_i^{\mathrm{p}} - \boldsymbol{a}_j^{\mathrm{p}}\right) \nabla W_{ij}}_{(\boldsymbol{A}\boldsymbol{p})_i} = \underbrace{\rho_0 - \rho_i^*}_{s_i} \qquad \boldsymbol{a}_i^{\mathrm{p}} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \nabla W_{ij}$$

– Interpretation:

$$\Delta t \sum_j m_j \left(\Delta t \boldsymbol{a}_i^{\mathrm{p}} - \Delta t \boldsymbol{a}_j^{\mathrm{p}}\right) \nabla W_{ij} = \rho_0 - \rho_i^*$$

$$\Delta t \sum_j m_j \left(\boldsymbol{v}_i^{\mathrm{p}} - \boldsymbol{v}_j^{\mathrm{p}}\right) \nabla W_{ij} = \rho_0 - \rho_i^*$$

$$\Delta t \cdot \rho_i \cdot \nabla \cdot \boldsymbol{v}_i^{\mathrm{p}} = \rho_0 - \rho_i^*$$

Pressure accelerations cause a velocity change v^p whose divergence causes a density change.

PPE Solver - IISPH

- Relaxed Jacobi: $p_i^{l+1} = \max\left(p_i^l + \omega \frac{s_i - (\boldsymbol{A}\boldsymbol{p}^l)_i}{a_{ii}}, 0\right)$
  - For IISPH, typically $\omega = 0.5$
  - Diagonal element a_{ii}
    - Accumulate all coefficients of p_i in $\Delta t^2 \sum_j m_j \left(\boldsymbol{a}_i^p - \boldsymbol{a}_j^p\right) \nabla W_{ij}$
    - $a_{ii} = \Delta t^2 \sum_j m_j \left(-\sum_j \frac{m_j}{\rho_j^2} \nabla W_{ij}\right) \cdot \nabla W_{ij} + \Delta t^2 \sum_j m_j \left(\frac{m_i}{\rho_i^2} \nabla W_{ji}\right) \cdot \nabla W_{ij}$

- Note, that the first pressure update is $p_i^1 = 0 + \omega \frac{s_i - 0}{a_{ii}} = \frac{\omega}{a_{ii}}(\rho_0 - \rho^*)$ State equation
- Using the incompressible PPE variant IISPH with one solver iteration corresponds to compressible state-equation SPH with $p_i = -\frac{\omega}{a_{ii}}(\rho^* - \rho_0)$

PPE Solver Implementation - IISPH

- Initialization: $\rho_i = \sum_j m_j W_{ij} \quad a_{ii} = \dots$

$$\boldsymbol{v}_i^* = \boldsymbol{v}_i + \Delta t \boldsymbol{a}_i^{\mathrm{nonp}}$$

$$s_i = \rho_0 - \rho_i - \Delta t \sum_j m_j \boldsymbol{v}_{ij}^* \nabla W_{ij}$$

$$p_i^0 = \max\left(\omega \frac{s_i}{a_{ii}}, 0\right)$$

- Solver update in iteration *l*:
  - First loop: $(\boldsymbol{a}_i^{\mathrm{p}})^l = -\sum_j m_j \left(\frac{p_i^l}{\rho_i^2} + \frac{p_j^l}{\rho_j^2}\right) \nabla W_{ij}$
  - Second loop: $(\boldsymbol{A}\boldsymbol{p}^l)_i = \Delta t^2 \sum_j m_j \left((\boldsymbol{a}_i^{\mathrm{p}})^l - (\boldsymbol{a}_j^{\mathrm{p}})^l\right) \nabla W_{ij}$

$$p_i^{l+1} = \max\left(p_i^l + \omega \frac{s_i - (\boldsymbol{A}\boldsymbol{p}^l)_i}{a_{ii}}, 0\right)$$ If a_{ii} not equal to zero

$$(\rho_i^{\mathrm{error}})^l = (\boldsymbol{A}\boldsymbol{p}^l)_i - s_i$$ Continue until error is small

Boundary Handling - IISPH

- PPE: $\Delta t^2 \nabla^2 p_f = \rho_0 - \rho_f^* = \rho_0 - \rho_f + \Delta t \rho_0 \nabla \cdot \boldsymbol{v}_f^*$
- Discretized PPE: $\boldsymbol{Ap} = \boldsymbol{s}$

Index f indicates a fluid sample.
Index b indicates a boundary sample.
f_f indicates a fluid neighbor of f.
f_b indicates a boundary neighbor of f.

$$(\boldsymbol{Ap})_f = \Delta t^2 \sum_{f_f} m_{f_f} \left(\boldsymbol{a}_f^{\mathrm{p}} - \boldsymbol{a}_{f_f}^{\mathrm{p}} \right) \nabla W_{f f_f} + \Delta t^2 \sum_{f_b} m_{f_b} \boldsymbol{a}_f^{\mathrm{p}} \nabla W_{f f_b}$$

$$\boldsymbol{a}_f^{\mathrm{p}} = -\sum_{f_f} m_{f_f} \left(\frac{p_f}{\rho_f^2} + \frac{p_{f_f}}{\rho_{f_f}^2} \right) \nabla W_{f f_f} - \gamma \sum_{f_b} m_{f_b} 2 \frac{p_f}{\rho_f^2} \nabla W_{f f_b}$$

$$s_f = \rho_0 - \rho_f - \Delta t \sum_{f_f} m_{f_f} \left(\boldsymbol{v}_f^* - \boldsymbol{v}_{f_f}^* \right) \nabla W_{f f_f} - \Delta t \sum_{f_b} m_{f_b} \left(\boldsymbol{v}_f^* - \boldsymbol{v}_{f_b}(t + \Delta t) \right) \nabla W_{f f_b}$$

Boundary Handling - IISPH

– Diagonal element

$$
\begin{aligned}
a_{ff} =& \Delta t^2 \sum_{f_f} m_{f_f} \left(-\sum_{f_f} \frac{m_{f_f}}{\rho_{f_f}^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{\rho_0^2} \nabla W_{f f_b} \right) \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_f} m_{f_f} \left(\frac{m_f}{\rho_f^2} \nabla W_{f_f f} \right) \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_b} m_{f_b} \left(-\sum_{f_f} \frac{m_{f_f}}{\rho_{f_f}^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{\rho_0^2} \nabla W_{f f_b} \right) \nabla W_{f f_b}
\end{aligned}
$$

IISPH with Boundary - Implementation

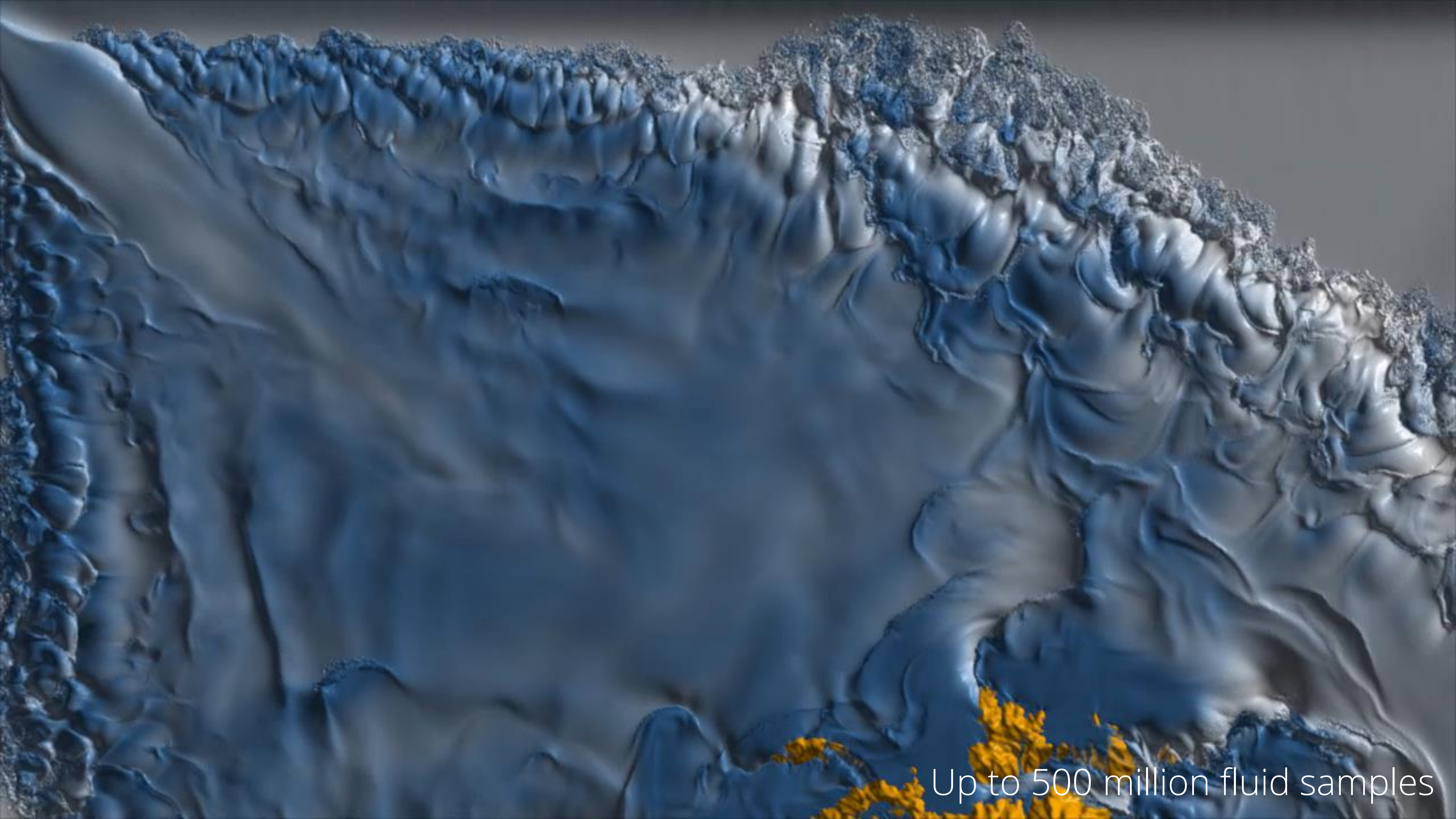
- Initialization:
  $$\rho_f = \sum_{f_f} m_{f_f} W_{ff_f} + \sum_{f_b} m_{f_b} W_{ff_b} \qquad a_{ff} = \dots$$
  $$\boldsymbol{v}_f^* = \boldsymbol{v}_f + \Delta t \boldsymbol{a}_f^{\text{nonp}}$$
  $$s_f = \rho_0 - \rho_f - \Delta t \sum_{f_f} m_{f_f} \boldsymbol{v}_{ff_f}^* \nabla W_{ff_f} - \Delta t \sum_{f_b} m_{f_b} \boldsymbol{v}_{ff_b}^* \nabla W_{ff_b}$$
  $$p_f^0 = \max\left(\omega \frac{s_f}{a_{ff}}, 0\right)$$
- Solver update in iteration *l*:
  - First loop:
    $$(\boldsymbol{a}_f^{\text{p}})^l = -\sum_{f_f} m_{f_f} \left(\frac{p_f^l}{\rho_f^2} + \frac{p_{f_f}^l}{\rho_{f_f}^2}\right) \nabla W_{ff_f} - \gamma \sum_{f_b} m_{f_b} 2 \frac{p_f^l}{\rho_f^2} \nabla W_{ff_b}$$
  - Second loop:
    $$(\boldsymbol{A}\boldsymbol{p}^l)_f = \Delta t^2 \sum_{f_f} m_{f_f} \left((\boldsymbol{a}_f^{\text{p}})^l - (\boldsymbol{a}_{f_f}^{\text{p}})^l\right) \nabla W_{ff_f} + \Delta t^2 \sum_{f_b} m_{f_b} (\boldsymbol{a}_f^{\text{p}})^l \nabla W_{ff_b}$$
    $$p_f^{l+1} = \max\left(p_f^l + \omega \frac{s_f - (\boldsymbol{A}\boldsymbol{p}^l)_f}{a_{ff}}, 0\right)$$
    If a_{ff} not equal to zero
    $$(\rho_f^{\text{error}})^l = (\boldsymbol{A}\boldsymbol{p}^l)_f - s_f$$
    Continue until error is small

IISPH vs. PCISPH

- Breaking dam
  - 100k samples with diameter 0.05m, 0.01% ave density error

	PCISPH			IISPH			PCISPH / IISPH		
		total comp. time [s]			total comp. time [s]		ratio		
Δt [s]	avg. iter.	pressure	overall	avg. iter.	pressure	overall	iterations	pressure	overall
0.0005	4.3	540	1195	2.2	148	978	2.0	3.6	1.2
0.00067	7.2	647	**1145**	2.9	149	753	2.5	4.3	1.5
0.001	14.9	856	1187	4.9	164	576	3.0	5.2	2.1
0.0025	66.5	1495	1540	18.4	242	410	3.6	**6.2**	3.8
0.004	-	-	-	33.5	273	**379**	-	-	-
0.005	-	-	-	45.8	297	383	-	-	-

- Largest possible time step does not necessarily result in the best performance


Up to 500 million fluid samples

Outline

- Introduction
- Concepts
  - State equation
  - Iterative state equation
  - Pressure Poisson equation
- Current developments

Current Developments

- DFSPH [Bender 2015]
  - Combination of two PPEs (inspired by [Hu 2007])
  - Resolving compressibility and removing velocity divergence in two steps
  - Currently the most efficient solver
- [Cornelis 2018]
  - Various formulations for combining two PPEs
- [Fuerstenau 2017]
  - Discretization of the Laplacian